THE EXTENSION OF $H^p$-FUNCTIONS FROM CERTAIN HYPERSURFACES OF A POLYDISC

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Abstract. Let $E$ be a subvariety of the open unit polydisc $U^n$, $n > 2$, of pure dimension $n - 1$, satisfying the following conditions. There exists an annular domain $Q^* = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : r < |z_i| < 1, 1 < i < n\}$, a continuous function $\eta : [r, 1) \to [r, 1)$, and a $\delta > 0$, such that

(i) $|z_n| < \eta((|z_1| + \cdots + |z_{n-1}|)/(n-1))$ whenever $(z_1, \ldots, z_n) \in E \cap Q^*$,

(ii) $|\alpha - \beta| > \delta$ whenever $1 < j < n$ and $(\zeta', \alpha, \zeta'') \neq (\zeta', \beta, \zeta'')$ are both in $(Q^{*-1} \times U \times Q^{*-1}) \cap E$.

Theorem. Let $0 < p < \infty$, let $g$ be holomorphic on $E$ and let $u$ be the real part of a holomorphic function on $E$. If $|g(z)|^p < u(z)$ for all $z \in E$, then $g$ can be extended to a function in the Hardy space $H^p(U^n)$.

In this article a set of conditions is given under which it is possible to extend $H^p$-functions from codimension-1 subvarieties of a polydisc. These conditions are essentially the same as those given by P. S. Chee ([2, Theorem 4.1, p. 111]) for the extension of $H^\infty$-functions, thereby providing a somewhat complete story in so far as all $p$, $0 < p < \infty$, is concerned.

The notation will be as in [2]. If $0 < r < 1$ then $U(r) = \{z \in \mathbb{C} : |z| < r\}$, if $0 < r < s$ then $Q(r, s) = \{z \in \mathbb{C} : r < |z| < s\}$. We write $U = U(1)$ and denote by $T$ its boundary, the unit circle. The cartesian product of $n$ copies of a set $S \subset \mathbb{C}$ will be represented by $S^n$, in particular, $U^n$ will be the open unit polydisc, and $T^n$ the unit $n$-torus. By a polydomain in $\mathbb{C}^n$ we mean a cartesian product of $n$ open connected subsets of $\mathbb{C}$.

Let $\Omega$ be a polydomain in $\mathbb{C}^n$ and let $p \in (0, \infty)$. The Hardy space $H^p(\Omega)$ consists of all holomorphic functions $f$ on $\Omega$ such that $|f|^p$ has an n-harmonic majorant on $\Omega$. We denote the class of bounded holomorphic functions on $\Omega$ by $H^\infty(\Omega)$.

Fix $\zeta_0 \in \Omega$. If $f \in H^p(\Omega)$, and if $u$ is the least n-harmonic majorant of $|f|^p$ on $\Omega$, we write

$$\|f\|_{H^p(\Omega)} = u(\zeta_0)^{1/p}.$$ 

As is well known, $\| \|_{H^p(\Omega)}$ endows $H^p(\Omega)$ with the structure of a Banach or Frechet space, depending on whether $1 < p < \infty$ or $0 < p < 1$. The topology of $H^p(\Omega)$ is stronger than that of local uniform convergence in $\Omega$. Furthermore, the choice of $\zeta_0$ is immaterial, for if we fix $p$ and vary $\zeta_0$, the corresponding "norms" define equivalent structures.
For the remainder of the paper, \( p \in (0, \infty) \) and \( n > 2 \) will be fixed.

Our first step is to prove \( H^p \) versions of Lemmas 1 and 2 of [1]. Fix \( 0 < r < 1 \) and write \( Q = Q(r, 1) \). If \( h \) is holomorphic on \( Q \) and has a Laurent expansion 
\[
h(z) = \sum_{m=-\infty}^{\infty} c(m)z^m,
\]
we define \( \Pi h(z) = \sum_{m=-\infty}^{\infty} c(m)z^m \). If \( h \) is holomorphic on \( Q^n \) and has a Laurent expansion
\[
h(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n} c(m_1, \ldots, m_n)z_1^{m_1} \cdots z_n^{m_n},
\]
we define \( \Pi_j h, 1 < j < n, \) to be the holomorphic function whose Laurent series is obtained by deleting above all terms in which \( m_j > 0 \).

1. **Lemma.** There exists a constant \( K \) such that
\[
\| \Pi h \|_{H^p(Q)} \leq K\| h \|_{H^p(Q)}
\]
for all \( h \in H^p(Q) \).

**Proof.** Clearly, \( \Pi \) is a continuous linear operator with respect to the topology of local uniform convergence in \( Q \). Also ([3, Theorem 10.12, p. 181]) \( \Pi h \in H^p(Q) \) whenever \( h \in H^p(Q) \). The Closed Graph Theorem then implies that \( \Pi \) is a bounded operator on \( H^p(Q) \), completing the proof.

2. **Lemma.** There exists a constant \( K \) such that
\[
\| \Pi_j h \|_{H^p(Q^n)} \leq K\| h \|_{H^p(Q^n)}
\]
for all \( h \in H^p(Q^n) \) and \( 1 < j < n \).

**Proof.** Fix \( z_0 \in Q \). Take \( z_0 \) and \( \xi_0 = (z_0, \ldots, z_0) \) as reference points for \( \| \|_{H^p(Q)} \) and \( \| \|_{H^p(Q^n)} \) respectively. Let \( \{ Q_k \} \) be an expanding sequence of annuli such that
(i) \( z_0 \in Q_k \),
(ii) \( Q_k \subset Q \),
(iii) \( Q = \bigcup Q_k \).

Let \( \Gamma_k \) be the positively oriented boundary of \( Q_k \), and let \( G_k(\cdot, z) \) be the Greens function of \( Q_k \) with pole at \( z \).

To prove our lemma we make the following observation. Let \( f \in H^p(Q^n) \), write \( \xi = (z_1, \ldots, z_n) \) and denote the exterior normal derivative by \( \partial / \partial \nu \). The \( n \)-harmonic functions
\[
u_k(\xi) = \left( \frac{1}{2\pi} \right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \ldots, w_n)|^p \frac{\partial}{\partial \nu} G_k(w_1, z_1) \cdots \frac{\partial}{\partial \nu} G_k(w_n, z_n) |dw_1| \cdots |dw_n|
\]
form an increasing sequence (since \( |f|^p \) is \( n \)-subharmonic), which, as can be easily seen, converges to the least \( n \)-harmonic majorant of \( |f|^p \) in \( Q^n \). Hence
\[
\| f \|_{H^p(Q^n)}^p = \sup_k \left( \frac{1}{2\pi} \right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \ldots, w_n)|^p \frac{\partial}{\partial \nu} G_k(w_1, z_0) \cdots \frac{\partial}{\partial \nu} G_k(w_n, z_0) |dw_1| \cdots |dw_n|.
\] (2.1)
Without loss of generality, set \( j = 1 \). Let \( u \) be the least \( n \)-harmonic majorant of \(|h|^p\) on \( Q^n \). Fix \( \xi' \in Q^{n-1} \), let \( \xi = (z_1, \xi') \in Q^n \) and define \( h_\xi(z_1) = h(\xi) \). Clearly \( h_\xi \in H^p(Q) \); in particular, \( u(\cdot, \xi') \) is a harmonic majorant of \(|h_\xi|^p\) on \( Q \). By Lemma 1,\
\[
\|\Pi h_\xi\|^p_{L^p(Q^*)} \leq K \|h_\xi\|^p_{L^p(Q)} \leq Ku(z_0, \xi')^{1/p}.
\] (2.2)
The relations (2.1), with \( n = 1 \) and \( f = \Pi h_\xi \), and (2.2), imply\
\[
\frac{1}{2\pi} \int_{\Gamma_k} |\Pi h_\xi(w_1)|^p \frac{\partial}{\partial \nu} G_k(z_0, w_1) \ |dw_1| < Ku(z_0, \xi').
\] (2.3)
Clearly \( \Pi h_\xi(z_1) = \Pi_1 h(\xi) \), so if we choose \( \xi' = (w_2, \ldots, w_n) \in \Gamma_k^{n-1} \), multiply both terms in (2.3) by \( \frac{\partial}{\partial \nu} G_k(z_0, w_2) \cdots \frac{\partial}{\partial \nu} G_k(z_0, w_n) \), and then integrate on \( \Gamma_k^{n-1} \) with respect to \( (\frac{1}{2\pi})^{n-1}|dw_2| \cdots |dw_n| \), we obtain\
\[
\left( \frac{1}{2\pi} \right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |\Pi_1 h(w_1, \ldots, w_n)|^p \frac{\partial}{\partial \nu} G_k(w_1, z_0) \cdots \\
\frac{\partial}{\partial \nu} G_k(w_n, z_0) |dw_1| \cdots |dw_n| < Ku(z_0, \xi').
\] (2.4)
Taking the supremum in (2.4) over all \( k \), we get\
\[
\|\Pi_1 h\|^p_{L^p(Q^*)} \leq Ku(z_0) = K^p\|h\|^p_{L^p(Q^*)},
\] which establishes the lemma.

The next lemmas, 3, 4 and 5, are listed for future reference; the proofs will be omitted. The proof of Lemma 3 is a straightforward generalization of the corresponding one-variable result (see the last paragraph on p. 182 of [3]). Lemmas 4 and 5 are proven in greater generality in [6] and [7].

Let \( V_j, 1 < j < n, \) be bounded domains in \( C \) with boundaries \( \partial V_j \). The distinguished boundary of \( U = V_1 \times \cdots \times V_n \) is the product \( \partial U = \partial V_1 \times \cdots \times \partial V_n \). We say that \( \partial U \) is analytic if each \( \partial V_j \) consists of finitely many disjoint closed analytic curves.

3. **Lemma.** Let \( \Omega \subset U \) be bounded polydomains in \( C^n \) with analytic distinguished boundaries \( \partial \Omega \subset \partial \Omega \). If \( f \) is holomorphic on \( \Omega \), and if its restriction to \( \Omega \) is in \( H^p(\Omega) \), then \( f \in H^p(\Omega) \).

For Lemmas 4 and 5, let \( \{\Omega_i\}_{i \in I} \) be a family of polydomains in \( C^n \) such that \( \overline{U^n} \subset \bigcup_{i \in I} \Omega_i \).

4. **Lemma** [6, Theorem 2.10, p. 301]. If \( f \) is holomorphic on \( U^n \) and if the restriction of \( f \) to each \( \Omega_i \cap U^n \) belongs to \( H^p(\Omega_i \cap U^n) \), then \( f \in H^p(U^n) \).

5. **Lemma** [7, Theorem 4.9]. For each \( i, j \in I \) let \( f_{ij} \in H^p(\Omega_i \cap \Omega_j \cap U^n) \) be given such that \( f_{ij} + f_{ik} + f_{jk} = 0 \) on any nonvoid intersection \( \Omega_i \cap \Omega_j \cap \Omega_k \cap U^n \). Then there exist functions \( f_i \in H^p(\Omega_i \cap U^n) \) such that \( f_j - f_i = f_{ij} \).

Let \( E \) be a subvariety of \( U^n \) of pure dimension \( n - 1 \) satisfying the following conditions. There exist \( r \in (0, 1) \), an annulus \( Q = Q(r, 1) \), a continuous function
\( \eta: [r, 1) \to [r, 1), \) and \( \delta > 0, \) such that
\[
|z_n| < \eta((|z_1| + \cdots + |z_{n-1}|)/(n - 1))
\]
whenever \( (z_1, \ldots, z_n) \in Q^n \cap E, \) and such that \( |\alpha - \beta| > \delta \) whenever \( 1 < j < n \)
and \( (\xi', \alpha, \xi'') \neq (\xi', \beta, \xi'') \) are in \( (Q^{n-1} \times U \times Q^{n-j}) \cap E. \)

6. Theorem. Let \( g \) be a holomorphic function on \( E, \) let \( u \) be a pluriharmonic
function on \( E, \) and assume \( |g(z)|^p < u(z) \) for all \( z \in E. \) Then \( g \) has an extension
\( G \in H^p(U^n). \)

Proof. The requirements on \( E \) imply, as is observed in [5] for the more
restrictive case \( \text{dist}(E, T^n) > 0, \) that \( (Q^{n-1} \times U) \cap E \) (and more generally any
product obtained by permuting the \( n \) factors) is an unbranched analytic cover of
\( Q^{n-1} \) of say \( m \) sheets. Thus, there are holomorphic functions \( \alpha_1, \ldots, \alpha_m \) on \( Q^{n-1} \)
such that
\[
(Q^{n-1} \times U) \cap E = \{(\xi', z_n) \in Q^{n-1} \times U: z_n = \alpha_j(\xi') \text{ for some } 1 < j < m\}.
\]
As in [5], define
\[
g_n(\xi') = \sum_{i=1}^{m} g(\xi', \alpha_i(\xi')) \prod_{1 < j < m} \frac{z_n - \alpha_j(\xi')}{\alpha_i(\xi') - \alpha_j(\xi')}, \tag{6.1}
\]
for \( \xi' = (\xi', z_n) \in Q^{n-1} \times U. \)

Clearly, \( g_n \) is holomorphic in \( Q^{n-1} \times U \) and agrees with \( g \) on \( (Q^{n-1} \times U) \cap E. \)
Since for each \( 1 < i < m \) the composition \( u_i(\xi') = u(\xi', \alpha_i(\xi')) \) is the real part of
some holomorphic function on \( Q^{n-1}, \) since
\[
|g(\xi', \alpha_i(\xi'))|^p < u_i(\xi'),
\]
and since \( |\alpha_i(\xi') - \alpha_j(\xi')| > \delta \) for \( i \neq j, \) it follows from (6.1) that \( |g_n|^p \)
\( g \) is majorized on \( Q^{n-1} \times U \) by the real part of a holomorphic function. In particular, \( g_n \in H^p(Q^{n-1} \times U). \)

A parallel construction to the above yields local extensions \( g_i \in H^p(Q^{i-1} \times U \times Q^{n-i}) \)
of \( g \) for each \( 1 < i < n. \)

By [2, Theorem 3.1, p. 110] there exists \( F \in H^\infty(U^n) \) such that \( E \) is the zero set
of \( F \) and such that \( F \) generates the ideal-sheaf of \( E. \) We define
\[
h_i = (\phi - g_i)/F, \tag{6.2}
\]
where \( \phi \) is a holomorphic extension of \( g \) on \( U^n \) (which exists by Cartan’s Theorem
B). Since \( F \) generates the ideal-sheaf of \( E, \) the functions \( h_i \) are well defined and
holomorphic on \( Q^{i-1} \times U \times Q^{n-i}. \)

To prove our theorem, we first consider the particular case \( \text{dist}(E, T^n) > 0. \)

By taking \( r \) larger, if necessary, we can assume \( \text{dist}(E, Q^n) > 0, \) and ([4,
Theorem 4.8.3, p. 91]) that \( 1/F \) is bounded on \( Q^n. \) This immediately implies
\( h_i - h_j = (g_j - g_i)/F \in H^p(Q^n) \) which with Lemma 2 and the fact that \( \Pi_j h_j = 0 \)
yields
\[
\Pi_j h_1 = \Pi_j (h_1 - h_j) \in H^p(Q^n). \tag{6.3}
\]
As in [1], we define

\[ h = (1 - \Pi_1)(1 - \Pi_2) \cdots (1 - \Pi_n)h_1, \tag{6.4} \]

and \( G = \phi - Fh. \) The function \( G \) is a holomorphic extension of \( g \) on \( U^n. \) We proceed to establish \( G \in H^p(U^n). \)

From (6.4) it follows that

\[ h - h_1 = -\sum_i \Pi_i h_1 + \sum_{i\neq j} \Pi_i \Pi_j h_1 - \cdots . \]

A repeated application of Lemma 2, together with (6.3), implies \( h - h_1 \in H^p(Q^n). \) This, and (6.2), gives us

\[ G = \phi - Fh = \phi - Fh_1 + F(h_1 - h) = g_1 + F(h_1 - h) \in H^p(Q^n). \]

Lemma 3 then implies \( G \in H^p(U^n). \)

We now consider the general case of the theorem.

Fix \( r' \in (r, 1), \) let

\[ c' = \sup \{ \eta(x): r < x < 1 - (1 - r')/(n - 1) \} , \]

and choose \( c \in (c', 1). \) Following [2] we define

\[ \mathcal{U}_i = U^{i-1} \times U(r') \times U^{n-i}, \quad 1 < i < n - 1, \]

\[ \mathcal{V}_n = Q^{n-1} \times U, \]

\[ \mathcal{W}_i = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times Q(c, 1), \quad 1 < i < n - 1. \]

We observe

\[ \mathcal{U}_i \cap \mathcal{U}_k = U^{i-1} \times U(r') \times U^{k-i-1} \times U(r') \times U^{n-k}, \quad 1 < i < k < n - 1, \]

\[ \mathcal{U}_i \cap \mathcal{V}_n = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times U, \quad 1 < i < n - 1. \]

Suppose \( 1 < i < n - 1. \) If \( (z_1, \ldots, z_{n-1}) \in Q^{i-1} \times Q(r,r') \times Q^{n-i-1} \) and \( (z_1, \ldots, z_{n-1}, z_n) \in E, \) then

\[ |z_n| < \eta((|z_1| + \cdots + |z_{n-1}|)/(n - 1)) < c' < c. \]

Hence \( \text{dist}(E, \mathcal{W}_i) > 0. \) We can then apply the special case of the theorem, proven above, to obtain extensions \( G_i \in H^p(\mathcal{U}_i) \) of \( g \) for \( 1 < i < n - 1. \)

In (6.1) we constructed an extension \( g_n \in H^p(\mathcal{U}_n) \) of \( g. \) We relabel \( g_n = G_n. \) The set of functions \( \{ G_i: 1 < i < n \} \) is then a complete set of local \( H^p \)-extensions of \( g. \)

Let \( 1 < i < j < n. \) Then \( G_i - G_j \in H^p(\mathcal{U}_i \cap \mathcal{U}_j), \) and \( G_i - G_j = 0 \) on \( \mathcal{U}_i \cap \mathcal{U}_j \cap E. \) Since \( F \) generates the ideal-sheaf of \( E, \) the functions

\[ f_{ij} = (G_i - G_j)/F \tag{6.5} \]

are well defined and holomorphic on \( \mathcal{U}_i \cap \mathcal{U}_j. \) Moreover, since \( 1/F \) is bounded on \( \mathcal{W}_i ([2, \text{Remark on p. 111}], \) we have \( f_{ij} \in H^p(\mathcal{W}_i \cap \mathcal{U}_i \cap \mathcal{U}_j). \) The functions \( f_{ij} \) are holomorphic on \( \mathcal{U}_i \cap \mathcal{U}_j, \) and the distinguished boundary of \( \mathcal{U}_i \cap \mathcal{U}_j \) is contained in that of \( \mathcal{W}_i \cap \mathcal{U}_i \cap \mathcal{U}_j. \) Lemma 3 then implies that \( f_{ij} \in H^p(\mathcal{W}_i \cap \mathcal{U}_j). \)

The sets \( \{ \mathcal{W}_i: 1 < i < n \} \) form an open cover of \( U^n. \) They can be enlarged to form an open cover of \( \bar{U}^n \) such that the intersection of the enlargement of \( \mathcal{U}_i \) with \( U^n \) is again \( \mathcal{U}_i. \) By Lemma 5 there exist functions \( f_{ij} \in H^p(\mathcal{U}_i) \) such that

\[ f_j - f_i = f_{ij}. \tag{6.6} \]
The functions $G_i + f_i F$ are in $H^p(_i)$ and extend $g$. Moreover, (6.5) and (6.6) imply $G_i + f_i F = G_j + f_j F$ on $_i \cap _j$. Hence we can analytically continue the functions $G_i + f_i F$ to a holomorphic function $G$ on $U^n$ which extends $g$. The restriction of $G$ to $_i$ (the function $G_i + f_i F$) is in $H^p(_i)$. Lemma 4 then implies $G \in H^p(U^n)$. This completes the proof.

REFERENCES


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