THE EXTENSION OF H^p-FUNCTIONS FROM CERTAIN HYPERSURFACES OF A POLYDISC

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ABSTRACT. Let $E$ be a subvariety of the open unit polydisc $U^n$, $n > 2$, of pure dimension $n-1$, satisfying the following conditions. There exists an annular domain $Q^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n: r < |z_i| < 1, 1 < i < n\}$, a continuous function $\eta: [r, 1) \to [r, 1)$, and a $\delta > 0$, such that

(i) $|z_n| < \eta((|z_1| + \cdots + |z_{n-1}|)/(n-1))$ whenever $(z_1, \ldots, z_n) \in E \cap Q^n$,
(ii) $|\alpha - \beta| > \delta$ whenever $1 < j < n$ and $(\xi', \alpha, \xi'') \neq (\xi', \beta, \xi'')$ are both in $(Q^{n-1} \times U \times Q^{n-1}) \cap E$.

Theorem. Let $0 < p < \infty$, let $g$ be holomorphic on $E$ and let $u$ be the real part of a holomorphic function on $E$. If $|g(z)|^p < u(z)$ for all $z \in E$, then $g$ can be extended to a function in the Hardy space $H^p(U^n)$.

In this article a set of conditions is given under which it is possible to extend $H^p$-functions from codimension-1 subvarieties of a polydisc. These conditions are essentially the same as those given by P. S. Chee ([2, Theorem 4.1, p. 111]) for the extension of $H^\infty$-functions, thereby providing a somewhat complete story in so far as all $p$, $0 < p < \infty$, is concerned.

The notation will be as in [2]. If $0 < r < 1$ then $U(r) = \{z \in \mathbb{C}: |z| < r\}$, if $0 < r < s$ then $Q(r, s) = \{z \in \mathbb{C}: r < |z| < s\}$. We write $U = U(1)$ and denote by $T$ its boundary, the unit circle. The cartesian product of $n$ copies of a set $S \subset \mathbb{C}$ will be represented by $S^n$, in particular, $U^n$ will be the open unit polydisc, and $T^n$ the unit $n$-torus. By a polydomain in $\mathbb{C}^n$ we mean a cartesian product of $n$ open connected subsets of $\mathbb{C}$.

Let $\Omega$ be a polydomain in $\mathbb{C}^n$ and let $p \in (0, \infty)$. The Hardy space $H^p(\Omega)$ consists of all holomorphic functions $f$ on $\Omega$ such that $|f|^p$ has an $n$-harmonic majorant on $\Omega$. We denote the class of bounded holomorphic functions on $\Omega$ by $H^\infty(\Omega)$.

Fix $\xi_0 \in \Omega$. If $f \in H^p(\Omega)$, and if $u$ is the least $n$-harmonic majorant of $|f|^p$ on $\Omega$, we write

$$||f||_{H^p(\Omega)} = u(\xi_0)^{1/p}.$$ 

As is well known, $|| \cdot ||_{H^p(\Omega)}$ endows $H^p(\Omega)$ with the structure of a Banach or Frechet space, depending on whether $1 < p < \infty$ or $0 < p < 1$. The topology of $H^p(\Omega)$ is stronger than that of local uniform convergence in $\Omega$. Furthermore, the choice of $\xi_0$ is immaterial, for if we fix $p$ and vary $\xi_0$, the corresponding "norms" define equivalent structures.

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For the remainder of the paper, $\rho \in (0, \infty)$ and $n \geq 2$ will be fixed.

Our first step is to prove $H^p$ versions of Lemmas 1 and 2 of [1]. Fix $0 < r < 1$ and write $Q = Q(r, 1)$. If $h$ is holomorphic on $Q$ and has a Laurent expansion $h(z) = \sum_{n=\infty}^{+\infty} c(m)z^m$, we define $\Pi h$ by $\Pi h(z) = \sum_{n=-\infty}^{-1} c(m)z^m$. If $h$ is holomorphic on $Q^n$ and has a Laurent expansion

$$h(z_1, \ldots, z_n) = \sum c(m_1, \ldots, m_n)z_1^{m_1}\cdots z_n^{m_n},$$

we define $\Pi_j h$, $1 < j < n$, to be the holomorphic function whose Laurent series is obtained by deleting above all terms in which $m_j > 0$.

1. **Lemma.** There exists a constant $K$ such that

$$\|\Pi h\|_{H^p(Q)} \leq K\|h\|_{H^p(Q)}$$

for all $h \in H^p(Q)$.

**Proof.** Clearly, $\Pi$ is a continuous linear operator with respect to the topology of local uniform convergence in $Q$. Also ([3, Theorem 10.12, p. 181]) $\Pi h \in H^p(Q)$ whenever $h \in H^p(Q)$. The Closed Graph Theorem then implies that $\Pi$ is a bounded operator on $H^p(Q)$, completing the proof.

2. **Lemma.** There exists a constant $K$ such that

$$\|\Pi_j h\|_{H^p(Q^n)} \leq K\|h\|_{H^p(Q^n)}$$

for all $h \in H^p(Q^n)$ and $1 < j < n$.

**Proof.** Fix $z_0 \in Q$. Take $z_0$ and $z_0 = (z_0, \ldots, z_0)$ as reference points for $\|h\|_{H^p(Q)}$ and $\|h\|_{H^p(Q^n)}$ respectively. Let $\{Q_k\}$ be an expanding sequence of annuli such that

(i) $z_0 \in Q_k$,
(ii) $Q_k \subset Q$,
(iii) $Q = \bigcup Q_k$.

Let $\Gamma_k$ be the positively oriented boundary of $Q_k$, and let $G_k(\cdot, z)$ be the Greens function of $Q_k$ with pole at $z$.

To prove our lemma we make the following observation. Let $f \in H^p(Q^n)$, write $f(z) = (z_1, \ldots, z_n)$ and denote the exterior normal derivative by $\partial / \partial v$. The $n$-harmonic functions

$$u_k(z) = \left(\frac{1}{2\pi}\right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \ldots, w_n)|^p \frac{\partial}{\partial v} G_k(w_1, z_1) \cdots$$

$$\frac{\partial}{\partial v} G_k(w_n, z_n) |dw_1| \cdots |dw_n|$$

form an increasing sequence (since $|f|^p$ is n-subharmonic), which, as can be easily seen, converges to the least $n$-harmonic majorant of $|f|^p$ in $Q^n$. Hence

$$\|f\|_{H^p(Q^n)} = \sup_k \left(\frac{1}{2\pi}\right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \ldots, w_n)|^p \frac{\partial}{\partial v} G_k(w_1, z_0) \cdots$$

$$\frac{\partial}{\partial v} G_k(w_n, z_0) |dw_1| \cdots |dw_n|.$$  \hfill (2.1)
Without loss of generality, set \( j = 1 \). Let \( u \) be the least \( n \)-harmonic majorant of \( |h|^p \) on \( Q^n \). Fix \( z' \in Q^{n-1} \), let \( z = (z_1, z') \in Q^n \) and define \( h_{z_1}(z_1) = h(z) \). Clearly \( h_{z_1} \in H^p(Q) \); in particular, \( u(\cdot, z') \) is a harmonic majorant of \( |h_{z_1}|^p \) on \( Q \). By Lemma 1,

\[
\|\Pi h_{z_1}\|_{H^p(Q)} \leq K \|h_{z_1}\|_{H^p(Q)} \leq Ku(z_0, z')^{1/p}.
\]

The relations (2.1), with \( n = 1 \) and \( f = \Pi h_{z_1} \), and (2.2), imply

\[
\frac{1}{2\pi} \int_{\Gamma_k} |\Pi h_{z_1}(w_1)| \frac{\partial}{\partial \nu} G_k(z_0, w_1) |dw_1| < K^p u(z_0, z').
\]

Clearly \( \Pi h_{z_1}(z_1) = \Pi_1 h(z') \), so if we choose \( z' = (w_2, \ldots, w_n) \in \Gamma_k^{n-1} \), multiply both terms in (2.3) by \( \frac{\partial}{\partial \nu} G_k(z_0, w_2) \cdots \frac{\partial}{\partial \nu} G_k(z_0, w_n) \), and then integrate on \( \Gamma_k^{n-1} \) with respect to \( (\frac{1}{2\pi})^{n-1} |dw_2| \cdots |dw_n| \), we obtain

\[
\left( \frac{1}{2\pi} \right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} \left| \Pi_1 h(w_1, \ldots, w_n) \right| \frac{\partial}{\partial \nu} G_k(w_1, z_0) \cdots \frac{\partial}{\partial \nu} G_k(w_n, z_0) |dw_1| \cdots |dw_n| < K^p u(z_0),
\]

Taking the supremum in (2.4) over all \( k \), we get

\[
\|\Pi_1 h\|_{H^p(Q')} < K^p u(z_0) = K^p \|h\|_{H^p(Q')},
\]

which establishes the lemma.

The next lemmas, 3, 4 and 5, are listed for future reference; the proofs will be omitted. The proof of Lemma 3 is a straightforward generalization of the corresponding one-variable result (see the last paragraph on p. 182 of [3]). Lemmas 4 and 5 are proven in greater generality in [6] and [7].

Let \( V_j, 1 < j < n, \) be bounded domains in \( C \) with boundaries \( \partial V_j \). The distinguished boundary of \( \mathcal{U} = V_1 \times \cdots \times V_n \) is the product \( \partial \mathcal{U} = \partial V_1 \times \cdots \times \partial V_n \). We say that \( \partial \mathcal{U} \) is analytic if each \( \partial V_j \) consists of finitely many disjoint analytic curves.

3. **Lemma.** Let \( \mathcal{D} \subset \mathcal{U} \) be bounded polydomains in \( C^n \) with analytic distinguished boundaries \( \text{d} \mathcal{U} \subset \text{d} \mathcal{D} \). If \( f \) is holomorphic on \( \mathcal{U} \), and if its restriction to \( \mathcal{D} \) is in \( H^p(\mathcal{D}) \), then \( f \in H^p(\mathcal{U}) \).

For Lemmas 4 and 5, let \( \{\mathcal{U}_i\}_{i \in I} \) be a family of polydomains in \( C^n \) such that \( \bar{U}^n \subset \bigcup_{i \in I} \mathcal{U}_i \).

4. **Lemma** [6, Theorem 2.10, p. 301]. If \( f \) is holomorphic on \( U^n \) and if the restriction of \( f \) to each \( \mathcal{U}_i \cap U^n \) belongs to \( H^p(\mathcal{U}_i \cap U^n) \), then \( f \in H^p(U^n) \).

5. **Lemma** [7, Theorem 4.9]. For each \( i, j \in I \) let \( f_{ij} \in H^p(\mathcal{U}_i \cap \mathcal{U}_j \cap U^n) \) be given such that \( f_{ij} + f_{ji} + f_{ki} = 0 \) on any nonvoid intersection \( \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap U^n \). Then there exist functions \( f_i \in H^p(\mathcal{U}_i \cap U^n) \) such that \( f_j - f_i = f_{ij} \).

Let \( E \) be a subvariety of \( U^n \) of pure dimension \( n - 1 \) satisfying the following conditions. There exist \( r \in (0, 1) \), an annulus \( Q = Q(r, 1) \), a continuous function
\[ \eta: [r, 1) \to [r, 1), \text{ and } \delta > 0, \text{ such that} \]
\[ |z_n| \leq \eta((|z_1| + \cdots + |z_{n-1}|)/(n-1)) \]
whenever \((z_1, \ldots, z_n) \in Q^n \cap E\), and such that \(|\alpha - \beta| > \delta \text{ whenever } 1 < j < n\)
and \((\zeta', \alpha, \zeta'') \neq (\zeta', \beta, \zeta'')\) are in \((Q^{n-1} \times U \times Q^{n-j}) \cap E\).

6. Theorem. Let \(g\) be a holomorphic function on \(E\), let \(u\) be a pluriharmonic function on \(E\), and assume \(|g(z)|^p < u(z)\) for all \(z \in E\). Then \(g\) has an extension \(G \in H^p(U^n)\).

Proof. The requirements on \(E\) imply, as is observed in [5] for the more restrictive case \(\text{dist}(E, T^n) > 0\), that \((Q^{n-1} \times U) \cap E\) (and more generally any product obtained by permuting the \(n\) factors) is an unbranched analytic cover of \(Q^{n-1}\) of say \(m\) sheets. Thus, there are holomorphic functions \(\alpha_1, \ldots, \alpha_m\) on \(Q^{n-1}\) such that

\[
(Q^{n-1} \times U) \cap E = \{ (\zeta', z_n) \in Q^{n-1} \times U: z_n = \alpha_j(\zeta') \text{ for some } 1 < j < m \}.
\]

As in [5], define

\[
g_n(\zeta) = \sum_{i=1}^{m} g(\zeta', \alpha_i(\zeta')) \prod_{1 < j < m} \frac{z_n - \alpha_j(\zeta')}{\alpha_i(\zeta') - \alpha_j(\zeta')} \tag{6.1}
\]

for \(\zeta = (\zeta', z_n) \in Q^{n-1} \times U\).

Clearly, \(g_n\) is holomorphic in \(Q^{n-1} \times U\) and agrees with \(g\) on \((Q^{n-1} \times U) \cap E\). Since for each \(1 < i < m\) the composition \(u_i(\zeta') = u(\zeta', \alpha_i(\zeta'))\) is the real part of some holomorphic function on \(Q^{n-1}\), since

\[ |g(\zeta', \alpha_i(\zeta'))|^p < u_i(\zeta'), \]

and since \(|\alpha_i(\zeta') - \alpha_j(\zeta')| > \delta\) for \(i \neq j\), it follows from (6.1) that \(|g_n|^p\) is majorized on \(Q^{n-1} \times U\) by the real part of a holomorphic function. In particular, \(g_n \in H^p(Q^{n-1} \times U)\).

A parallel construction to the above yields local extensions \(g_i \in H^p(Q^{i-1} \times U \times Q^{n-i})\) of \(g\) for each \(1 < i < n\).

By [2, Theorem 3.1, p. 110] there exists \(F \in H^\infty(U^n)\) such that \(E\) is the zero set of \(F\) and such that \(F\) generates the ideal-sheaf of \(E\). We define

\[
h_j = (\phi - g_i) / F, \tag{6.2}
\]

where \(\phi\) is a holomorphic extension of \(g\) on \(U^n\) (which exists by Cartan’s Theorem B). Since \(F\) generates the ideal-sheaf of \(E\), the functions \(h_i\) are well defined and holomorphic on \(Q^{i-1} \times U \times Q^{n-i}\).

To prove our theorem, we first consider the particular case \(\text{dist}(E, T^n) > 0\).

By taking \(r\) larger, if necessary, we can assume \(\text{dist}(E, Q^n) > 0\), and ([4, Theorem 4.8.3, p. 91]) that \(1/F\) is bounded on \(Q^n\). This immediately implies \(h_i - h_j = (g_j - g_i)/F \in H^p(Q^n)\) which with Lemma 2 and the fact that \(\Pi_j h_j = 0\) yields

\[
\Pi_j h_1 = \Pi_j (h_1 - h_j) \in H^p(Q^n). \tag{6.3}
\]
As in [1], we define
\[ h = (1 - \Pi_1)(1 - \Pi_2) \cdots (1 - \Pi_n)h, \quad (6.4) \]
and \( G = \phi - Fh \). The function \( G \) is a holomorphic extension of \( g \) on \( U^n \). We proceed to establish \( G \in H^p(U^n) \).

From (6.4) it follows that
\[ h - h = -\Sigma \Pi_i h + \Sigma \pi_j \Pi_i \Pi_j h = + \cdots. \]
A repeated application of Lemma 2, together with (6.3), implies \( h - h \in H^p(Q^n) \).

This, and (6.2), gives us
\[ G = \phi - Fh = \phi - Fh + F(h - h) = g_1 + F(h_1 - h) \in H^p(Q^n). \]

Lemma 3 then implies \( G \in H^p(U^n) \).

We now consider the general case of the theorem.

Fix \( r' \in (r, 1) \), let
\[ c' = \sup \{ \eta(x) : r < x < 1 - (1 - r')/(n - 1) \}, \]
and choose \( c \in (c', 1) \). Following [2] we define
\[ \mathcal{U}_i = U^{i-1} \times U(r') \times U^{n-i}, \quad 1 < i < n - 1, \]
\[ \mathcal{U}_n = Q^{n-1} \times U, \]
\[ \mathcal{Q}_i = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times Q(c, 1), \quad 1 < i < n - 1. \]

We observe
\[ \mathcal{U}_i \cap \mathcal{U}_k = U^{i-1} \times U(r') \times U^{k-i-1} \times U(r') \times U^{n-k}, \quad 1 < i < k < n - 1, \]
\[ \mathcal{U}_i \cap \mathcal{U}_n = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times U, \quad 1 < i < n - 1. \]

Suppose \( 1 < i < n - 1 \). If \( (z_1, \ldots, z_{n-1}) \in Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \) and \( (z_1, \ldots, z_{n-1}, z_n) \in E \), then
\[ \left| \frac{z_1}{\eta}(1 \leq j \leq n) \right| < c' < c. \]
Hence \( \text{dist}(E, \mathcal{Q}_i) > 0 \). We can then apply the special case of the theorem, proven above, to obtain extensions \( G_i \in H^p(\mathcal{Q}_i) \) of \( g \) for \( 1 < i < n - 1 \).

In (6.1) we constructed an extension \( g_n \in H^p(\mathcal{U}_n) \) of \( g \). We relabel \( g_n = G_n \). The set of functions \( \{ G_i : 1 < i < n \} \) is then a complete set of local \( H^p \)-extensions of \( g \).

Let \( 1 < i < j < n \). Then \( G_i - G_j \in H^p(\mathcal{U}_i \cap \mathcal{U}_j) \), and \( G_i - G_j = 0 \) on \( \mathcal{U}_i \cap \mathcal{U}_j \cap E \). Since \( F \) generates the ideal-sheaf of \( E \), the functions
\[ f_{ij} = \frac{(G_i - G_j)}{F} \quad (6.5) \]
are well defined and holomorphic on \( \mathcal{U}_i \cap \mathcal{U}_j \). Moreover, since \( 1/F \) is bounded on \( \mathcal{Q}_i \), we have \( f_{ij} \in H^p(\mathcal{Q}_i \cap \mathcal{U}_i \cap \mathcal{U}_j) \). The functions \( f_{ij} \) are holomorphic on \( \mathcal{U}_i \cap \mathcal{U}_j \), and the distinguished boundary of \( \mathcal{U}_i \cap \mathcal{U}_j \) is contained in that of \( \mathcal{Q}_i \cap \mathcal{U}_i \cap \mathcal{U}_j \). Lemma 3 then implies that \( f_{ij} \in H^p(\mathcal{Q}_i \cap \mathcal{Q}_j) \).

The sets \( \mathcal{U}_i : 1 < i < n \) form an open cover of \( U^n \). They can be enlarged to form an open cover of \( U^n \) such that the intersection of the enlargement of \( \mathcal{U}_i \) with \( U^n \) is again \( \mathcal{U}_i \). By Lemma 5 there exist functions \( f_i \in H^p(\mathcal{U}_i) \) such that
\[ f_j - f_i = f_{ij}. \quad (6.6) \]
The functions $G_i + f_i F$ are in $H^p(\mathcal{U}_i)$ and extend $g$. Moreover, (6.5) and (6.6) imply $G_i + f_i F = G_j + f_j F$ on $\mathcal{U}_i \cap \mathcal{U}_j$. Hence we can analytically continue the functions $G_i + f_i F$ to a holomorphic function $G$ on $U^n$ which extends $g$. The restriction of $G$ to $\mathcal{U}_i$ (the function $G_i + f_i F$) is in $H^p(\mathcal{U}_i)$. Lemma 4 then implies $G \in H^p(U^n)$. This completes the proof.

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