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THE EXTENSION OF H^p-FUNCTIONS FROM CERTAIN HYPERSURFACES OF A POLYDISC

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ABSTRACT. Let E be a subvariety of the open unit polydisc U^n, $n \geqslant 2$, of pure dimension $n - 1$, satisfying the following conditions. There exists an annular domain $Q^n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n: r < |z_i| < 1, 1 \leqslant i \leqslant n\}$, a continuous function $\eta: [r, 1) \to [r, 1)$, and a $\delta > 0$, such that

(i) $|z_n| \leqslant \eta((|z_1| + \cdots + |z_{n-1}|)/(n-1))$ whenever $(z_1, \ldots, z_n) \in E \cap Q^n$,

(ii) $|\alpha - \beta| \geqslant \delta$ whenever $1 \leqslant j \leqslant n$ and $(\zeta', \alpha, \zeta'') \neq (\zeta', \beta, \zeta'')$ are both in $(Q^{j-1} \times U \times Q^{n-j}) \cap E$.

THEOREM. *Let $0 < p < \infty$, let g be holomorphic on E and let u be the real part of a holomorphic function on E. If $|g(z)|^p \leqslant u(z)$ for all $z \in E$, then g can be extended to a function in the Hardy space $H^p(U^n)$.*

In this article a set of conditions is given under which it is possible to extend H^p-functions from codimension-1 subvarieties of a polydisc. These conditions are essentially the same as those given by P. S. Chee ([**2**, Theorem 4.1, p. 111]) for the extension of H^∞-functions, thereby providing a somewhat complete story in so far as all p, $0 \leqslant p \leqslant \infty$, is concerned.

The notation will be as in [**2**]. If $0 < r \leqslant 1$ then $U(r) = \{z \in \mathbf{C}: |z| < r\}$, if $0 < r < s$ then $Q(r, s) = \{z \in \mathbf{C}: r < |z| < s\}$. We write $U = U(1)$ and denote by T its boundary, the unit circle. The cartesian product of n copies of a set $S \subset \mathbf{C}$ will be represented by S^n, in particular, U^n will be the open unit polydisc, and T^n the unit n-torus. By a polydomain in $\mathbf{C}^n$ we mean a cartesian product of n open connected subsets of $\mathbf{C}$.

Let Ω be a polydomain in $\mathbf{C}^n$ and let $p \in (0, \infty)$. The Hardy space $H^p(\Omega)$ consists of all holomorphic functions f on Ω such that $|f|^p$ has an n-harmonic majorant on Ω. We denote the class of bounded holomorphic functions on Ω by $H^\infty(\Omega)$.

Fix $\zeta_0 \in \Omega$. If $f \in H^p(\Omega)$, and if u is the least n-harmonic majorant of $|f|^p$ on Ω, we write

$$\|f\|_{H^p(\Omega)} = u(\zeta_0)^{1/p}.$$

As is well known, $\| \ \|_{H^p(\Omega)}$ endows $H^p(\Omega)$ with the structure of a Banach or Frechet space, depending on whether $1 \leqslant p < \infty$ or $0 < p < 1$. The topology of $H^p(\Omega)$ is stronger than that of local uniform convergence in Ω. Furthermore, the choice of ζ_0 is immaterial, for if we fix p and vary ζ_0, the corresponding "norms" define equivalent structures.

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For the remainder of the paper, $p \in (0, \infty)$ and $n \geq 2$ will be fixed.

Our first step is to prove H^p versions of Lemmas 1 and 2 of [1]. Fix $0 < r < 1$ and write $Q = Q(r, 1)$. If h is holomorphic on Q and has a Laurent expansion $h(z) = \sum_{-\infty}^{+\infty} c(m)z^m$, we define Πh by $\Pi h(z) = \sum_{-\infty}^{-1} c(m)z^m$. If h is holomorphic on Q^n and has a Laurent expansion

$$h(z_1, \dots, z_n) = \sum c(m_1, \dots, m_n) z_1^{m_1} \cdots z_n^{m_n},$$

we define $\Pi_j h$, $1 \leq j \leq n$, to be the holomorphic function whose Laurent series is obtained by deleting above all terms in which $m_j \geq 0$.

1. Lemma. *There exists a constant K such that*

$$\|\Pi h\|_{H^p(Q)} \leq K\|h\|_{H^p(Q)}$$

for all $h \in H^p(Q)$.

Proof. Clearly, Π is a continuous linear operator with respect to the topology of local uniform convergence in Q. Also ([3, Theorem 10.12, p. 181]) $\Pi h \in H^p(Q)$ whenever $h \in H^p(Q)$. The Closed Graph Theorem then implies that Π is a bounded operator on $H^p(Q)$, completing the proof.

2. Lemma. *There exists a constant K such that*

$$\|\Pi_j h\|_{H^p(Q^n)} \leq K\|h\|_{H^p(Q^n)}$$

for all $h \in H^p(Q^n)$ *and* $1 \leq j \leq n$.

Proof. Fix $z_0 \in Q$. Take z_0 and $\zeta_0 = (z_0, \dots, z_0)$ as reference points for $\| \ \|_{H^p(Q)}$ and $\| \ \|_{H^p(Q^n)}$ respectively. Let $\{Q_k\}$ be an expanding sequence of annulli such that

(i) $z_0 \in Q_k$,
(ii) $\overline{Q}_k \subset Q$,
(iii) $Q = \bigcup Q_k$.

Let Γ_k be the positively oriented boundary of Q_k, and let $G_k(\cdot, z)$ be the Greens function of Q_k with pole at z.

To prove our lemma we make the following observation. Let $f \in H^p(Q^n)$, write $\zeta = (z_1, \dots, z_n)$ and denote the exterior normal derivative by $\partial/\partial\nu$. The n-harmonic functions

$$u_k(\zeta) = \left(\frac{1}{2\pi}\right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \dots, w_n)|^p \frac{\partial}{\partial\nu} G_k(w_1, z_1) \cdots \frac{\partial}{\partial\nu} G_k(w_n, z_n) |dw_1| \cdots |dw_n|$$

form an increasing sequence (since $|f|^p$ is n-subharmonic), which, as can be easily seen, converges to the least n-harmonic majorant of $|f|^p$ in Q^n. Hence

$$\|f\|_{H^p(Q^n)}^p = \sup_k \left(\frac{1}{2\pi}\right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |f(w_1, \dots, w_n)|^p \frac{\partial}{\partial\nu} G_k(w_1, z_0) \cdots \frac{\partial}{\partial\nu} G_k(w_n, z_0) |dw_1| \cdots |dw_n|. \tag{2.1}$$

Without loss of generality, set $j = 1$. Let u be the least n-harmonic majorant of $|h|^p$ on Q^n. Fix $\zeta' \in Q^{n-1}$, let $\zeta = (z_1, \zeta') \in Q^n$ and define $h_{\zeta'}(z_1) = h(\zeta)$. Clearly $h_{\zeta'} \in H^p(Q)$; in particular, $u(\cdot, \zeta')$ is a harmonic majorant of $|h_{\zeta'}|^p$ on Q. By Lemma 1,

$$\|\Pi h_{\zeta'}\|_{H^p(Q)} \leq K\|h_{\zeta'}\|_{H^p(Q)} \leq Ku(z_0, \zeta')^{1/p}. \tag{2.2}$$

The relations (2.1), with $n = 1$ and $f = \Pi h_{\zeta'}$, and (2.2), imply

$$\frac{1}{2\pi}\int_{\Gamma_k} |\Pi h_{\zeta'}(w_1)|^p \frac{\partial}{\partial \nu} G_k(z_0, w_1)\, |dw_1| \leq K^p u(z_0, \zeta'). \tag{2.3}$$

Clearly $\Pi h_{\zeta'}(z_1) = \Pi_1 h(\zeta)$, so if we choose $\zeta' = (w_2, \ldots, w_n) \in \Gamma_k^{n-1}$, multiply both terms in (2.3) by $\frac{\partial}{\partial \nu} G_k(z_0, w_2) \cdots \frac{\partial}{\partial \nu} G_k(z_0, w_n)$, and then integrate on Γ_k^{n-1} with respect to $(\frac{1}{2\pi})^{n-1}|dw_2| \cdots |dw_n|$, we obtain

$$\left(\frac{1}{2\pi}\right)^n \int_{\Gamma_k} \cdots \int_{\Gamma_k} |\Pi_1 h(w_1, \ldots, w_n)|^p \frac{\partial}{\partial \nu} G_k(w_1, z_0) \cdots \frac{\partial}{\partial \nu} G_k(w_n, z_0)|dw_1| \cdots |dw_n| \leq K^p u(\zeta_0). \tag{2.4}$$

Taking the supremum in (2.4) over all k, we get

$$\|\Pi_1 h\|^p_{H^p(Q^n)} \leq K^p u(\zeta_0) = K^p \|h\|^p_{H^p(Q^n)},$$

which establishes the lemma.

The next lemmas, 3, 4 and 5, are listed for future reference; the proofs will be omitted. The proof of Lemma 3 is a straightforward generalization of the corresponding one-variable result (see the last paragraph on p. 182 of [3]). Lemmas 4 and 5 are proven in greater generality in [6] and [7].

Let V_j, $1 \leq j \leq n$, be bounded domains in $\mathbf{C}$ with boundaries ∂V_j. The *distinguished boundary* of $\mathfrak{U} = V_1 \times \cdots \times V_n$ is the product $\check{\partial}\mathfrak{U} = \partial V_1 \times \cdots \times \partial V_n$. We say that $\check{\partial}\mathfrak{U}$ is *analytic* if each ∂V_j consists of finitely many disjoint closed analytic curves.

3. Lemma. *Let $\mathfrak{Q} \subset \mathfrak{U}$ be bounded polydomains in $\mathbf{C}^n$ with analytic distinguished boundaries $\check{\partial}\mathfrak{U} \subset \check{\partial}\mathfrak{Q}$. If f is holomorphic on $\mathfrak{U}$, and if its restriction to $\mathfrak{Q}$ is in $H^p(\mathfrak{Q})$, then $f \in H^p(\mathfrak{U})$.*

For Lemmas 4 and 5, let $\{\mathfrak{U}_i\}_{i \in I}$ be a family of polydomains in $\mathbf{C}^n$ such that $\bar{U}^n \subset \bigcup_{i \in I} \mathfrak{U}_i$.

4. Lemma [6, Theorem 2.10, p. 301]. *If f is holomorphic on U^n and if the restriction of f to each $\mathfrak{U}_i \cap U^n$ belongs to $H^p(\mathfrak{U}_i \cap U^n)$, then $f \in H^p(U^n)$.*

5. Lemma [7, Theorem 4.9]. *For each $i, j \in I$ let $f_{ij} \in H^p(\mathfrak{U}_i \cap \mathfrak{U}_j \cap U^n)$ be given such that $f_{ij} + f_{jk} + f_{ki} = 0$ on any nonvoid intersection $\mathfrak{U}_i \cap \mathfrak{U}_j \cap \mathfrak{U}_k \cap U^n$. Then there exist functions $f_i \in H^p(\mathfrak{U}_i \cap U^n)$ such that $f_j - f_i = f_{ij}$.*

Let E be a subvariety of U^n of pure dimension $n - 1$ satisfying the following conditions. There exist $r \in (0, 1)$, an annulus $Q = Q(r, 1)$, a continuous function

$\eta\colon [r, 1) \to [r, 1)$, and $\delta > 0$, such that

$$|z_n| \leqslant \eta((|z_1| + \cdots + |z_{n-1}|)/(n-1))$$

whenever $(z_1, \ldots, z_n) \in Q^n \cap E$, and such that $|\alpha - \beta| \geqslant \delta$ whenever $1 \leqslant j \leqslant n$ and $(\zeta', \alpha, \zeta'') \neq (\zeta', \beta, \zeta'')$ are in $(Q^{j-1} \times U \times Q^{n-j}) \cap E$.

6. THEOREM. *Let g be a holomorphic function on E, let u be a pluriharmonic function on E, and assume* $|g(z)|^p \leqslant u(z)$ *for all* $z \in E$. *Then g has an extension* $G \in H^p(U^n)$.

PROOF. The requirements on E imply, as is observed in [5] for the more restrictive case $\operatorname{dist}(E, T^n) > 0$, that $(Q^{n-1} \times U) \cap E$ (and more generally any product obtained by permuting the n factors) is an unbranched analytic cover of Q^{n-1} of say m sheets. Thus, there are holomorphic functions $\alpha_1, \ldots, \alpha_m$ on Q^{n-1} such that

$$(Q^{n-1} \times U) \cap E = \left\{(\zeta', z_n) \in Q^{n-1} \times U\colon z_n = \alpha_j(\zeta') \text{ for some } 1 \leqslant j \leqslant m\right\}.$$

As in [5], define

$$g_n(\zeta) = \sum_{i=1}^{m} g(\zeta', \alpha_i(\zeta')) \prod_{\substack{i \neq j \\ 1 \leqslant j \leqslant m}} \frac{z_n - \alpha_j(\zeta')}{\alpha_i(\zeta') - \alpha_j(\zeta')} \tag{6.1}$$

for $\zeta = (\zeta', z_n) \in Q^{n-1} \times U$.

Clearly, g_n is holomorphic in $Q^{n-1} \times U$ and agrees with g on $(Q^{n-1} \times U) \cap E$. Since for each $1 \leqslant i \leqslant m$ the composition $u_i(\zeta') = u(\zeta', \alpha_i(\zeta'))$ is the real part of some holomorphic function on Q^{n-1}, since

$$|g(\zeta', \alpha_i(\zeta'))|^p \leqslant u_i(\zeta'),$$

and since $|\alpha_i(\zeta') - \alpha_j(\zeta')| \geqslant \delta$ for $i \neq j$, it follows from (6.1) that $|g_n|^p$ is majorized on $Q^{n-1} \times U$ by the real part of a holomorphic function. In particular, $g_n \in H^p(Q^{n-1} \times U)$.

A parallel construction to the above yields local extensions $g_i \in H^p(Q^{i-1} \times U \times Q^{n-i})$ of g for each $1 \leqslant i \leqslant n$.

By [2, Theorem 3.1, p. 110] there exists $F \in H^\infty(U^n)$ such that E is the zero set of F and such that F generates the ideal-sheaf of E. We define

$$h_i = (\phi - g_i)/F, \tag{6.2}$$

where ϕ is a holomorphic extension of g on U^n (which exists by Cartan's Theorem B). Since F generates the ideal-sheaf of E, the functions h_i are well defined and holomorphic on $Q^{i-1} \times U \times Q^{n-i}$.

To prove our theorem, we first consider the particular case $\operatorname{dist}(E, T^n) > 0$.

By taking r larger, if necessary, we can assume $\operatorname{dist}(E, Q^n) > 0$, and ([4, Theorem 4.8.3, p. 91]) that $1/F$ is bounded on Q^n. This immediately implies $h_i - h_j = (g_j - g_i)/F \in H^p(Q^n)$ which with Lemma 2 and the fact that $\Pi_j h_j = 0$ yields

$$\Pi_j h_1 = \Pi_j(h_1 - h_j) \in H^p(Q^n). \tag{6.3}$$

As in [1], we define

$$h = (1 - \Pi_1)(1 - \Pi_2) \cdots (1 - \Pi_n)h_1, \tag{6.4}$$

and $G = \phi - Fh$. The function G is a holomorphic extension of g on U^n. We proceed to establish $G \in H^p(U^n)$.

From (6.4) it follows that

$$h - h_1 = -\Sigma_i \Pi_i h_1 + \Sigma_{i \neq j} \Pi_i \Pi_j h_1 - + \cdots .$$

A repeated application of Lemma 2, together with (6.3), implies $h - h_1 \in H^p(Q^n)$. This, and (6.2), gives us

$$G = \phi - Fh = \phi - Fh_1 + F(h_1 - h) = g_1 + F(h_1 - h) \in H^p(Q^n).$$

Lemma 3 then implies $G \in H^p(U^n)$.

We now consider the general case of the theorem.

Fix $r' \in (r, 1)$, let

$$c' = \sup\{\eta(x): r \leq x \leq 1 - (1 - r')/(n - 1)\},$$

and choose $c \in (c', 1)$. Following [2] we define

$$\mathfrak{U}_i = U^{i-1} \times U(r') \times U^{n-i}, \qquad 1 \leq i \leq n - 1,$$
$$\mathfrak{U}_n = Q^{n-1} \times U,$$
$$\mathfrak{Q}_i = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times Q(c, 1), \qquad 1 \leq i \leq n - 1.$$

We observe

$$\mathfrak{U}_i \cap \mathfrak{U}_k = U^{i-1} \times U(r') \times U^{k-i-1} \times U(r') \times U^{n-k}, \qquad 1 \leq i < k \leq n - 1,$$
$$\mathfrak{U}_i \cap \mathfrak{U}_n = Q^{i-1} \times Q(r, r') \times Q^{n-i-1} \times U, \qquad 1 \leq i \leq n - 1.$$

Suppose $1 \leq i \leq n - 1$. If $(z_1, \ldots, z_{n-1}) \in Q^{i-1} \times Q(r, r') \times Q^{n-i-1}$ and $(z_1, \ldots, z_{n-1}, z_n) \in E$, then

$$|z_n| \leq \eta((|z_1| + \cdots + |z_{n-1}|)/(n - 1)) \leq c' < c.$$

Hence $\operatorname{dist}(E, \mathfrak{Q}_i) > 0$. We can then apply the special case of the theorem, proven above, to obtain extensions $G_i \in H^p(\mathfrak{U}_i)$ of g for $1 \leq i \leq n - 1$.

In (6.1) we constructed an extension $g_n \in H^p(\mathfrak{U}_n)$ of g. We relabel $g_n = G_n$. The set of functions $\{G_i: 1 \leq i \leq n\}$ is then a complete set of local H^p-extensions of g.

Let $1 \leq i < j \leq n$. Then $G_i - G_j \in H^p(\mathfrak{U}_i \cap \mathfrak{U}_j)$, and $G_i - G_j = 0$ on $\mathfrak{U}_i \cap \mathfrak{U}_j \cap E$. Since F generates the ideal-sheaf of E, the functions

$$f_{ij} = (G_i - G_j)/F \tag{6.5}$$

are well defined and holomorphic on $\mathfrak{U}_i \cap \mathfrak{U}_j$. Moreover, since $1/F$ is bounded on $\mathfrak{Q}_i$ ([2, Remark on p. 111]), we have $f_{ij} \in H^p(\mathfrak{Q}_i \cap \mathfrak{U}_i \cap \mathfrak{U}_j)$. The functions f_{ij} are holomorphic on $\mathfrak{U}_i \cap \mathfrak{U}_j$, and the distinguished boundary of $\mathfrak{U}_i \cap \mathfrak{U}_j$ is contained in that of $\mathfrak{Q}_i \cap \mathfrak{U}_i \cap \mathfrak{U}_j$. Lemma 3 then implies that $f_{ij} \in H^p(\mathfrak{U}_i \cap \mathfrak{U}_j)$.

The sets $\{\mathfrak{U}_i: 1 \leq i \leq n\}$ form an open cover of U^n. They can be enlarged to form an open cover of $\overline{U}^n$ such that the intersection of the enlargement of $\mathfrak{U}_i$ with U^n is again $\mathfrak{U}_i$. By Lemma 5 there exist functions $f_i \in H^p(\mathfrak{U}_i)$ such that

$$f_j - f_i = f_{ij}. \tag{6.6}$$

The functions $G_i + f_i F$ are in $H^p(\mathfrak{U}_i)$ and extend g. Moreover, (6.5) and (6.6) imply $G_i + f_i F = G_j + f_j F$ on $\mathfrak{U}_i \cap \mathfrak{U}_j$. Hence we can analytically continue the functions $G_i + f_i F$ to a holomorphic function G on U^n which extends g. The restriction of G to $\mathfrak{U}_i$ (the function $G_i + f_i F$) is in $H^p(\mathfrak{U}_i)$. Lemma 4 then implies $G \in H^p(U^n)$. This completes the proof.

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