VECTOR VALUED INEQUALITIES FOR FOURIER SERIES

JOSÉ L. RUBIO DE FRANCIA

ABSTRACT. Denoting by S^* the maximal partial sum operator of Fourier series, we prove that $S^*(f_1, f_2, \ldots, f_k, \ldots) = (S^*f_1, S^*f_2, \ldots, S^*f_k, \ldots)$ is a bounded operator from $L^p(l^r)$ to itself, $1 < p, r < \infty$. Thus, we extend the theorem of Carleson and Hunt on pointwise convergence of Fourier series to the case of vector valued functions. We give also an application to the rectangular convergence of double Fourier series.

Let $S_n f$ denote the nth partial sum of the Fourier series of the function $f \in L^1(T) = L^1([-\pi, \pi))$. The maximal operator $S^*f = \sup_n |S_n f|$ satisfies the inequality of Carleson and Hunt

$$\|S^*f\|_p \leqslant C_p \|f\|_p \qquad (f \in L^p, 1 < p < \infty) \tag{1}$$

(see [5]). The following vector valued extension of (1) holds.

THEOREM. *If* $1 < p, r < \infty$ *and* (f_k) *is a sequence of functions in* $L^p(T)$

$$\left\| \left(\sum_k [S^*f_k]^r \right)^{1/r} \right\|_p \leqslant C_{p,r} \left\| \left(\sum_k |f_k|^r \right)^{1/r} \right\|_p.$$

When $p = r$ the result is obvious from (1). When $r < p$ we shall derive it from the weighted norm inequality for Fourier series obtained by R. Hunt and W.-S. Young [6] in the same way followed by A. Córdoba and C. Fefferman [3] for the case of singular integral operators. The case $p < r$ is not a consequence of the preceding one since S^* is by no means a selfadjoint operator, but a duality argument can still be used since a certain dual of the weighted norm inequality in [6] also holds.

We recall that u satisfies Muckenhoupt's condition A_p (see [8]) with constant K if, for all intervals I,

$$m_I(u) m_I(u^{-p'/p})^{p/p'} \leqslant K \tag{2}$$

where $m_I(\cdot)$ stands for the mean value over I. It is shown in [6] that if (2) holds

$$\int [S^*f]^p u \leqslant B \int |f|^p u \tag{3}$$

where B is constant depending only on K and p.

Assume first that $r < p$, and let $p/r = q$. Given the sequence (f_k), there is a function $u \in L^{q'}$ with $\|u\|_{q'} = 1$ such that

$$\int \left(\sum_k \left[S^* f_k \right]^r \right)^{p/r} = \left\{ \int \sum_k |S^* f_k|^r u \right\}^q \leqslant \left\{ \sum_k \int \left[S^* f_k \right]^r u_s \right\}^q$$

where s is a fixed number with $1 < s < q'$ and u_s is defined by

$$u_s(x) = \sup_{x \in I} \left(\frac{1}{|I|} \int_I |u|^s \right)^{1/s} = \left[(u^s)^* \right]^{1/s} \tag{4}$$

(by g^* we denote the Hardy-Littlewood maximal function of g). It is known that u_s satisfies A_1, and hence A_r, with constant K depending only on s (see [3]). Thus

$$\int \left(\sum_k |S^* f_k|^r \right)^{p/r} \leqslant B_{r,s} \left\{ \sum_k \int |f_k|^r u_s \right\}^q$$

$$\leqslant B_{r,s} \|u_s\|_{q'}^q \int \left(\sum_k |f_k|^r \right)^q.$$

But $u \mapsto u_s$ is a bounded operator in $L^{q'}(T)$ and the result follows.

Now we consider the case $p < r$. It will suffice to prove the theorem for the operator $S_N^* f = \sup_{n \leqslant N} |S_n f|$ with a constant $C_{p,r}$ independent of N. Given a finite collection $\alpha = \{h_1, h_2, \ldots, h_N\}$ of measurable functions with $\sum_1^N |h_k(x)| \leqslant 1$, we define the operator $T_\alpha f = \sum_{k=1}^N S_k(f h_k)$ and we assert the following

$$\left\| \left(\sum_k |T_{\alpha_k} f_k|^s \right)^{1/s} \right\|_q \leqslant C_{q,s} \left\| \left(\sum_k |f_k|^s \right)^{1/s} \right\|_q \qquad (1 < s < q < \infty) \tag{5}$$

where each α_k is as above and $C_{q,s}$ depends only on q and s.

For each $f \in L^1$ there is some $\alpha = \{h_1, \ldots, h_N\}$ such that, for every $g \in L^1$,

$$\int (S_N^* f) g = \int f(T_\alpha g)$$

(define $h_k = \operatorname{sgn}(S_k f)$ on A_k and $h_k = 0$ outside of A_k, where $\{A_1, \ldots, A_N\}$ is a partition of T such that $S_N^* f = |S_k f|$ on A_k). Since $r' < p'$ the theorem follows from (5) and the duality between $L^p(l^r)$ and $L^{p'}(l^{r'})$.

If u satisfies A_q with constant K $(1 < q < \infty)$, we shall prove that

$$\int |T_\alpha f|^q u \leqslant B \int |f|^q u \tag{6}$$

where B depends on K and q only. This is based on (3) and the fact that u satisfies A_q with constant K if and only if $u^{-q'/q}$ satisfies $A_{q'}$ with constant $K^{q'/q}$. Thus, for an adequate $w \in L^{q'}$ with $\|w\|_{q'} = 1$

$$\int |T_\alpha f|^q u = \left\{ \int (T_\alpha f) u^{1/q} w \right\}^q$$

$$= \left\{ \int f \sum_k h_k S_k(u^{1/q} w) \right\}^q \leqslant \left\{ \int |f| S^*(u^{1/q} w) \right\}^q$$

$$\leqslant \left\{ \int |f|^q u \right\} \left\{ \int u^{-q'/q} S^*(u^{1/q} w)^{q'} \right\}^{q/q'}$$

$$\leqslant \left\{ \int |f|^q u \right\} B \left\{ \int u^{-q'/q} |u^{1/q} w|^{q'} \right\}^{q/q'} = B \int |f|^q u.$$

We derive (5) from (6) by the same argument used in the case $r < p$, and the proof is ended.

COROLLARY 1. *Let* $f = (f_k) \in L^p(l^r)$, $1 < p$, $r < \infty$, *and let* $S_n f(x) = (S_n f_1(x), S_n f_2(x), \ldots, S_n f_k(x), \ldots)$ *be the partial sums of the (vector valued) Fourier series of* f. *Then*

$$\lim_n \|S_n f(x) - f(x)\|_{l^r} = 0 \quad (a.e.).$$

The proof is as usual, since the result is true for the functions f with a finite number of nonvanishing components, and these are dense in $L^p(l^r)$.

We can obviously replace l^r by L^r in the preceding corollary, and since $L^p(L^r(T))$ is identified with the Benedek-Panzone space $L^{r,p}(T^2)$ (see [1]) we obtain

COROLLARY 2. *Let* $f \in L^{r,p}(T^2)$ *with* $1 < r, p < \infty$. *If* $(S_{nm}f)_{n,m}$ *are the rectangular sums of the Fourier series of* f, *then*

$$\lim_{n,m} \|S_{nm}f(\cdot, y) - f(\cdot, y)\|_r = 0 \quad (a.e.\ y \in T).$$

In fact, by the preceding remark, we know that the function

$$g_m(x, y) = \sum_{k=-m}^{m} \sum_{j=-\infty}^{\infty} \hat{f}(j, k) e^{i(jx + ky)}$$

satisfies, for almost every $y \in T$

$$\lim_m \| g_m(\cdot, y) - f(\cdot, y)\|_r = 0. \tag{7}$$

If we fix any $y \in T$ for which (7) holds,

$$\lim_n \|S_{nm}f(\cdot, y) - g_m(\cdot, y)\|_r = \lim_n \|S_n g_m(\cdot, y) - g_m(\cdot, y)\|_r = 0$$

and the convergence is uniform in m because $\{ g_m(\cdot, y) | m \in N \}$ is a precompact subset of $L^p(T)$, so that

$$\lim_{n,m} S_{nm}f(\cdot, y) = \lim_m \lim_n S_{nm}f(\cdot, y)$$

$$= f(\cdot, y) \quad (\text{in } L^r\text{-norm}).$$

REMARK. (i) Both corollaries are in fact consequences of the maximal inequality

$$\left\| \left(\sup_n \sum_k |S_n f_k|^r \right)^{1/r} \right\|_p \leqslant C_{p,r} \left\| \left(\sum_k |f_k|^r \right)^{1/r} \right\|_p \tag{8}$$

which is weaker than the one we have actually proved. We observe that (8) is very easy when $2p/(p + 1) < r < 2p$ (by interpolation between the trivial case $p = r$ and the case $r = 2$ which is a consequence of the theorem of Marcinkiewicz and Zygmund [7]), but is not obvious for the remaining values of r and p.

(ii) Our Corollary 2 is in some sense intermediate between the positive result

$$\lim_{n,m} \|S_{nm}f - f\|_{r,p} = 0 \quad (f \in L^{r,p}(T^2))$$

(a consequence of M. Riesz's theorem) and the failure of pointwise convergence of the double sequence $S_{nm}f(x, y)$ which was proved by C. Fefferman [4].

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FACULTAD DE CIENCIAS, UNIVERSIDAD DE ZARAGOZA, ZARAGOZA, SPAIN