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PERTURBATIONS OF GROUND STATES OF TYPE I C^*-ALGEBRAS

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ABSTRACT. It is shown that the class of irreducible representations of a type I C^*-algebra A which satisfy a spectrum condition for a given dynamical system on A is unchanged if the system undergoes a sufficiently small relatively bounded perturbation. It follows that if A is also unital, then the existence of ground states is unaffected by such perturbations.

Let (A, α) be a C^*-dynamical system, consisting of a C^*-algebra A and a strongly continuous homomorphism $t \to \alpha_t$ of $\mathbf{R}$ into the group of $*$-automorphisms of A. The infinitesimal generator δ_α of α is a closed $*$-derivation of A defined on a dense $*$-subalgebra $\mathfrak{D}_\alpha$ of A, and the general theory of contraction semigroups **[7]**, **[12]** ensures that if δ_0 is any bounded $*$-derivation of A, then $\delta_\alpha + \delta_0$ is the generator of some dynamical system on A (a *bounded perturbation* of α). Longo **[9]** showed that any $*$-derivation δ_1 of A whose domain contains $\mathfrak{D}_\alpha$ is δ_α-bounded in the sense that there are constants β_1 and β_2 such that

$$\|\delta_1(a)\| \leqslant \beta_1\|a\| + \beta_2\|\delta_\alpha(a)\| \qquad (a \in \mathfrak{D}_\alpha)$$

and it was shown in **[3]** that $\delta_\alpha + \delta_1$ is the generator of a dynamical system on A whenever $\beta_2 < 1$. Slightly more generally, if δ and δ' are any two $*$-derivations with the same domain $\mathfrak{D}$ satisfying $\|(\delta' - \delta)(a)\| \leqslant \beta_1\|a\| + \beta_2(\|\delta(a)\| + \|\delta'(a)\|)$ for all a in $\mathfrak{D}$, for some $\beta_2 < 1$, then δ' is the (pre)generator of some C^*-dynamical system (A, α') if and only if δ is the (pre)generator of some system (A, α) **[12]**. In this event, α and α' (and also δ and δ') will be said to be *small perturbations* of each other (so any bounded perturbation is a small perturbation with $\beta_2 = 0$).These results establish the framework for perturbation theory of C^*-dynamical systems and their applications to statistical mechanics.

A state ϕ of A is said to be an *α-ground state* if ϕ annihilates the right ideal $R^\alpha(0, \infty)A$, where $R^\alpha(0, \infty)$ is the spectral subspace of α introduced by Arveson **[2]** (see also **[10]**). The α-ground states are α-invariant and form a weak*-closed face of the state space of A. Furthermore an arbitrary α-invariant state ϕ with associated cyclic representation $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ and covariant unitary representation u of $\mathbf{R}$ on $\mathcal{H}_\phi$ satisfying

$$u_t(\pi_\phi(a)\xi_\phi) = \pi_\phi(\alpha_t(a))\xi_\phi$$

is a ground state if and only if $u_t = \exp(ith)$ for some positive selfadjoint operator h on $\mathcal{H}_\phi$ with $h\xi_\phi = 0$ **[11]**. Thus ground states correspond to states of minimum

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energy in the C^*-algebraic model of statistical mechanics.

Powers and Sakai [11] showed that any approximately inner unital C^*-dynamical system possesses a ground state, but Lance and Niknam [8] constructed a simple system without one. This raised the problem of determining which systems do possess ground states. Let $(\mathcal{H}, \pi)$ be any representation of A which is α-covariant in the sense that α is implemented by some essentially selfadjoint operator h on $\mathcal{H}$, so that

$$\pi(\alpha_t(a)) = \exp(it\bar{h})\pi(a)\exp(-it\bar{h})$$

(where $\bar{h}$ is the closure of h). If h can be chosen to be lower semibounded, then $(\mathcal{H}, \pi)$ is said to *satisfy a spectrum condition* for α. In this case, if λ is the infimum of the spectrum of h, and unit vectors ξ_n $(n = 1, 2, \dots)$ in $\mathcal{H}$ are chosen so that $\|h\xi_n - \lambda\xi_n\| \to 0$, then any weak*-limit point of the states $\phi_n(a) = \langle\pi(a)\xi_n, \xi_n\rangle$ is an α-ground state [13]. Thus a unital C^*-dynamical system has a ground state if and only if it has a nondegenerate representation satisfying a spectrum condition. If α' is a bounded perturbation of α, then the same representations of A satisfy spectrum conditions for α' as for α. Thus the existence of ground states in unital C^*-algebras is invariant under bounded perturbations of the dynamical system [6], [13]. Here it will be shown that if A is also of type I and α' is any small perturbation of α, then the same irreducible representations of A satisfy spectrum conditions for α' as for α, and hence the existence of ground states is again invariant. Since the C^*-algebras of interest in quantum statistical mechanics are usually simple, this restriction limits the physical significance of the result, but it may point the way to further developments of the theory.

PROPOSITION 1. *Let (A, α) be a C^*-dynamical system, J be a closed α-invariant ideal in A, π be the quotient map of A onto A/J, and $(A/J, \tilde{\alpha})$ be the C^*-dynamical system satisfying $\pi \circ \alpha = \tilde{\alpha} \circ \pi$.*

(i) *$\mathfrak{D}_{\tilde{\alpha}} = \pi(\mathfrak{D}_\alpha)$, and for a in $\mathfrak{D}_\alpha$ and $\varepsilon > 0$, $\delta_{\tilde{\alpha}}(\pi(a)) = \pi(\delta_\alpha(a))$ and there exists b in $\mathfrak{D}_\alpha \cap J$ such that*

$$\|a - b\| < \|\pi(a)\| + \varepsilon,$$
$$\|\delta_\alpha(a - b)\| < \|\pi(\delta_\alpha(a))\| + \varepsilon.$$

(ii) *J is invariant under any small perturbation α' of α.*

PROOF. (i) It is clear that $\mathfrak{D}_{\tilde{\alpha}}$ contains $\pi(\mathfrak{D}_\alpha)$, and $\delta_{\tilde{\alpha}}(\pi(a)) = \pi(\delta_\alpha(a))$. Furthermore $\pi(\mathfrak{D}_\alpha)$ is $\tilde{\alpha}$-invariant and dense in A/J, so it is a core for $\delta_{\tilde{\alpha}}$ [5, Theorem 3].

Let f be a nonnegative continuously differentiable function on $\mathbf{R}$ with support in $[-1, 1]$ such that $\int_{-1}^{1} f(t)\, dt = 1$, and let

$$\eta = 2\varepsilon^{-1}\|a\|\int_{-1}^{1}|f'(t)|\, dt.$$

For a' in A and an increasing approximate unit $(u_i)_{i\in I}$ for J, $\|\alpha_t(a')(1 - u_i)\|$ decreases to $\|\pi(\alpha_t(a'))\| = \|\pi(a')\|$, and it follows from Dini's theorem that the convergence is uniform for t in compact subsets of $\mathbf{R}$. Hence for a in $\mathfrak{D}_\alpha$, u may be chosen in J so that $0 \leqslant u \leqslant 1$ and for t in $[-\eta, \eta]$,

$$\|\alpha_t(a)(1-u)\| < \|\pi(a)\| + \varepsilon,$$
$$\|\alpha_t(\delta_\alpha(a))(1-u)\| < \|\pi(\delta_\alpha(a))\| + \tfrac{1}{2}\varepsilon.$$

Let $b = \int_{-1}^{1} f(t) a\alpha_{\eta t}(u)\,dt$. Then b belongs to $\mathfrak{D}_\alpha \cap J$, and

$$\begin{aligned}\|a-b\| &\leq \int_{-1}^{1} f(t)\|a(1-\alpha_{\eta t}(u))\|\,dt\\ &= \int_{-1}^{1} f(t)\|\alpha_{-\eta t}(a)(1-u)\|\,dt\\ &< \|\pi(a)\| + \varepsilon,\end{aligned}$$

$$\begin{aligned}\|\delta_\alpha(a-b)\| &= \left\|\delta_\alpha(a) - \int_{-1}^{1} f(t)\delta_\alpha(a)\alpha_{\eta t}(u)\,dt + \frac{1}{\eta}\int_{-1}^{1} f'(t) a\alpha_{\eta t}(u)\,dt\right\|\\ &\leq \int_{-1}^{1} f(t)\|\alpha_{-\eta t}(\delta_\alpha(a))(1-u)\|\,dt + \frac{\|a\|}{\eta}\int_{-1}^{1}|f'(t)|\,dt\\ &< \|\pi(\delta_\alpha(a))\| + \varepsilon.\end{aligned}$$

It follows from the above estimates that the algebraic isomorphism between $\mathfrak{D}_\alpha/\mathfrak{D}_\alpha \cap J$ and $\pi(\mathfrak{D}_\alpha)$ is an isometry when $\mathfrak{D}_\alpha$ is equipped with the δ_α-graph norm, $\mathfrak{D}_\alpha/\mathfrak{D}_\alpha \cap J$ with the quotient norm and $\pi(\mathfrak{D}_\alpha)$ with the $\delta_{\tilde{\alpha}}$-graph norm. Since $\mathfrak{D}_\alpha/\mathfrak{D}_\alpha \cap J$ is a Banach space, the restriction of $\delta_{\tilde{\alpha}}$ to $\pi(\mathfrak{D}_\alpha)$ is closed. Since $\pi(\mathfrak{D}_\alpha)$ is also a core for $\delta_{\tilde{\alpha}}$, it is the whole domain.

(ii) Since δ_α maps $\mathfrak{D}_\alpha \cap J$ into J, it follows from [3, Lemma 2, Corollary 5] that $\mathfrak{D}_\alpha \cap J$ is dense in J, and $\delta_{\alpha'}$ also maps $\mathfrak{D}_\alpha \cap J$ into J. The restriction $\delta_{\alpha'}|_J$ of $\delta_{\alpha'}$ to $\mathfrak{D}_\alpha \cap J$ is a *-derivation of J, and is a small perturbation of $\delta_\alpha|_J$. Since $\delta_\alpha|_J$ is the generator of the C^*-dynamical system $(J, \alpha|_J)$, it follows from [3, Corollary 7], [12, Theorem X.50] that $\delta_{\alpha'}|_J$ is the generator of some C^*-dynamical system α'' on J. It is clear (e.g. by considering analytic vectors) that $\alpha_t''(a) = \alpha_t'(a)$ for a in J, so J is α'-invariant.

Proposition 2. *Let (A, α) be a C^*-dynamical system, where A is of type* I, *and let $(\mathcal{H}, \pi)$ be an irreducible representation of A whose kernel is α-invariant.*

(i) *There is an operator b in $\mathfrak{D}_\alpha$ and a unit vector ξ in $\mathcal{H}$ such that $\pi(b)$ is the projection of $\mathcal{H}$ onto $\mathbf{C}\xi$.*

(ii) *$(\mathcal{H}, \pi)$ is α-covariant.*

Let b and ξ be as in (i) *and h be a selfadjoint operator on $\mathcal{H}$ implementing α.*

(iii) *h is uniquely determined up to the addition of a scalar multiple λ of the identity.*

(iv) *h is essentially selfadjoint on $\pi(\mathfrak{D}_\alpha)\xi$, and λ may be chosen so that*

$$ih\pi(a)\xi = \pi(\delta_\alpha(ab))\xi = \pi(\delta_\alpha(ab^2))\xi \qquad (a \in \mathfrak{D}_\alpha).$$

Proof. Since π is irreducible and A is of type I, $\pi(A)$ contains all compact operators on $\mathcal{H}$, so (i) follows from [4, Theorem 8] and Proposition 1. If h and h' are selfadjoint operators on implementing α, then $\exp(-ith')\exp(ith)$ commutes with $\pi(A)$, and is therefore a scalar multiple of the identity. Part (iii) of the proposition follows. To prove (ii) and (iv), it now suffices to take fixed b and ξ as in (i), and construct h implementing α and satisfying (iv).

Let $b_1 = i(b\delta_\alpha(b) - \delta_\alpha(b)b)$, and $\delta'(a) = \delta_\alpha(a) - i(b_1 a - ab_1)$ $(a \in \mathfrak{D}_\alpha)$. The proof of [4, Theorem 8] shows that $b\delta_\alpha(b)b$, $\delta'(b)$ and $b\delta'(a)b$ all belong to the kernel of π, so the state $a \to \langle \pi(a)\xi, \xi\rangle$ is invariant under the bounded perturbation α' of α generated by δ'. Hence α' is implemented by the essentially selfadjoint operator h' with domain $\pi(\mathfrak{D}_\alpha)\xi$, given by

$$ih'\pi(a)\xi = \pi(\delta'(a))\xi \qquad (a \in \mathfrak{D}_\alpha).$$

Let h be the closure of $h' + \pi(b_1)$. Standard arguments (see [13, Proposition 9.6]) show that h implements α. Furthermore

$$\begin{aligned} ih\pi(a)\xi &= \pi(\delta_\alpha(a))\xi + i\pi(ab_1)\xi \\ &= \pi(\delta_\alpha(a)b)\xi - \pi(ab\delta_\alpha(b))\xi + \pi(a\delta_\alpha(b)b)\xi \\ &= \pi(\delta_\alpha(ab))\xi \end{aligned}$$

since $\pi(b\delta_\alpha(b)b) = 0$. But $\pi(\delta_\alpha(ab^2 - ab)b) = \pi(ab\delta_\alpha(b)b) = 0$.

THEOREM 3. *Let (A, α) and (A, α') be C^*-dynamical systems which are small perturbations of each other, where A is of type* I. *The same irreducible representations of A satisfy spectrum conditions for α as for α'.*

PROOF. For simplicity, let $\delta = \delta_\alpha$, $\delta' = \delta_{\alpha'}$, and $\mathfrak{D}$ be the common domain of δ and δ'. By symmetry, it suffices to show that any irreducible representation $(\mathcal{H}, \pi)$ satisfying a spectrum condition for α also satisfies one for α'. Standard iterative techniques (see the proof of [12, Theorem X.13]) enable us to assume that

$$\|(\delta' - \delta)(a)\| \leqslant \beta_1\|a\| + \beta_2\|\delta(a)\| \qquad (a \in \mathfrak{D})$$

for some $\beta_2 < 1$.

By Proposition 1, the kernel of π is α'-invariant, and by Proposition 2, there exist a unit vector ξ in $\mathcal{H}$, an operator b in $\mathfrak{D}$ and essentially selfadjoint operators h and h' on $\pi(\mathfrak{D})\xi$, implementing α and α' respectively, with h lower semibounded, such that $\pi(b)$ is the projection of $\mathcal{H}$ onto $\mathbf{C}\xi$, and for a in $\mathfrak{D}$,

$$\begin{aligned} ih\pi(a)\xi &= \pi(\delta(ab))\xi = \pi(\delta(ab^2))\xi, \\ ih'\pi(a)\xi &= \pi(\delta'(ab))\xi = \pi(\delta'(ab^2))\xi. \end{aligned}$$

Then

$$\begin{aligned} \|(h' - h)\pi(a)\xi\| &\leqslant \|\pi((\delta' - \delta)(ab^2))\| \\ &\leqslant \inf\{\|(\delta' - \delta)(ab^2 - a')\| : a' \in \mathfrak{D}, \pi(a') = 0\} \\ &\leqslant \inf\{\beta_1\|ab^2 - a'\| + \beta_2\|\delta(ab^2 - a')\| : a' \in \mathfrak{D}, \pi(a') = 0\} \\ &= \beta_1\|\pi(ab^2)\| + \beta_2\|\pi(\delta(ab^2))\| \\ &\leqslant \beta_1\|\pi(ab)\| + \beta_2\|\pi(ab\delta(b))\| + \beta_2\|\pi(\delta(ab)b)\| \\ &\leqslant (\beta_1 + \beta_2\|\delta(b)\|)\|\pi(ab)\| + \beta_2\|\pi(\delta(ab)b)\| \\ &= (\beta_1 + \beta_2\|\delta(b)\|)\|\pi(a)\xi\| + \beta_2\|h\pi(a)\xi\| \end{aligned}$$

where part (i) of Proposition 1 has been used in the fourth line. Since h and h' are essentially selfadjoint on $\pi(\mathfrak{D})\xi$, it follows from the Kato-Rellich theorem [**7**, Theorem 5.4.11], [**12**, Theorem X.12] that h' is lower semibounded, so π satisfies a spectrum condition for α'.

COROLLARY 4. *Let (A, α) be a C^*-dynamical system with a ground state, where A is unital and of type* I. *Then any small perturbation of α also has a ground state.*

PROOF. Since the α-ground states form a weak*-closed face of the state space of A, it follows from the Krein-Milman theorem that there is a pure state ϕ of A which is an α-ground state. Then π_ϕ is an irreducible representation of A satisfying a spectrum condition for α and hence for any small perturbation. The existence of ground states for small perturbations now follows from [**13**, Proposition 6.10].

It seems possible that Theorem 3 and therefore Corollary 4 may be valid for C^*-algebras which are not of type I. However the above methods depend heavily on Proposition 2, even part (ii) of which is a strong result. It is not difficult to see that for any C^*-dynamical system (A, α) and pure state ϕ of A, the following three conditions are equivalent:

(i) in the representation $(\mathcal{H}_\phi, \pi_\phi)$, α is implemented by a selfadjoint operator whose domain contains ξ_ϕ,

(ii) there is a bounded perturbation α' of α such that ϕ is α'-invariant,

(iii) there is a constant β such that for all selfadjoint a in $\mathfrak{D}_\alpha$,

$$|\phi(\delta_\alpha(a))|^2 \leqslant \beta\phi(a^2).$$

Furthermore, for any pure state ψ, $(\mathcal{H}_\psi, \pi_\psi)$ is α-covariant if and only if ψ is a norm-limit of pure states ϕ satisfying (i) to (iii). Thus any more general result would seem to require new techniques of proof. Since the spectrum condition can be characterised in terms of the spectral subspaces $R^\alpha(t, \infty)$ [**10**, Proposition 3.5.3], a possible alternative approach to that given above would be to consider the behaviour of $R^\alpha(t, \infty)$ under small perturbations, but this does not seem to lead to a proof of either result.

Under bounded perturbations, the KMS states at nonzero temperatures show greater stability than ground states. In particular there is a one-to-one correspondence between the KMS states of the original and perturbed systems [**1**], [**6**, Corollary 5.4.5]. One might therefore expect that there would be an analogue of Corollary 4 for KMS states, but, if so, this would also appear to require new techniques of proof.

REFERENCES

1. H. Araki, *Relative Hamiltonian for faithful normal states of a von Neumann algebra*, Publ. Res. Inst. Math. Sci. **9** (1973), 165–209.

2. W. B. Arveson, *On groups of automorphisms of operator algebras*, J. Functional Analysis **15** (1974), 217–243.

3. C. J. K. Batty, *Small perturbations of C*-dynamical systems*, Comm. Math. Phys. **68** (1978), 39–43.

4. O. Bratteli and D. W. Robinson, *Unbounded derivations of C*-algebras*, Comm. Math. Phys. **42** (1975), 253–268.

5. ______, *Unbounded derivations of C*-algebras*. II, Comm. Math. Phys. **46** (1976), 11–30.

6. ______, *Operator algebras and quantum statistical mechanics*. II, Springer, Berlin (to appear).

7. T. Kato, *Perturbation theory for linear operators*, 2nd ed., Springer, Berlin, 1976.

8. E. C. Lance and A. Niknam, *Unbounded derivations of group C^*-algebras*, Proc. Amer. Math. Soc. **6** (1976), 310–314.

9. R. Longo, *Automatic relative boundedness of derivations in C^*-algebras*, J. Functional Analysis **34** (1979), 21–28.

10. D. Olesen, *On spectral subspaces and their applications to automorphism groups*, Symposia Mathematica, vol. 20, Academic Press, New York, 1976, pp. 253–296.

11. R. T. Powers and S. Sakai, *Existence of ground states and KMS states for approximately inner dynamics*, Comm. Math. Phys. **39** (1975), 273–288.

12. M. Reed and B. Simon, *Methods of modern mathematical physics*. II, Academic Press, New York, 1975.

13. S. Sakai, *The theory of unbounded derivations in C^*-algebras*, Newcastle Univ. Lecture Notes, 1977.

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