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CHARACTERIZATION OF THE TRACE-CLASS

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ABSTRACT. We characterize the trace-class $\tau(A)$ associated with an H^*-algebra A as well as the trace-class (τc) of operators acting on a Hilbert space.

In this note we present a simple characterization of the trace-class $\tau(A)$ associated with an H^*-algebra A. An interesting special case of this result is a characterization of the trace-class (τc) [4, p. 36] of operators acting on a Hilbert space. To the best of our knowledge this is the first time a characterization of this class has been established.

An important role in the characterization is played by the property stated in the following lemma.

LEMMA 1. *Let A be a proper H^*-algebra* [1] *and let $\tau(A)$ be its trace-class* [5]. *Then the norm $\tau(\ )$ of $\tau(A)$ has the following property* $(x \in \tau(A))$: $(*)$ $\tau(x) = \text{lub}\{|\text{tr}(ax)|: a \in \tau(A)$ *and* $\text{lub}\{\text{tr}(y^*a^*ay): y \in \tau(A), \tau(y^*y) \leq 1\} \leq 1\}$.

PROOF. This is a consequence of the Lemma on p. 101 of [6] if we would take into account the fact that the set of the right centralizers of the form La: $x \to ax$ with $a \in \tau(A)$ is dense in the space $C(A)$ (defined on p. 101 of [6]) and that $\|La\| = \{\text{lub tr}(y^*a^*ay): a \in \tau(A), \tau(y^*y) \leq 1\}$.

Our characterization is based on the notion of a trace-algebra, which we are about to define.

DEFINITION. A Banach algebra B with the norm $n(\ )$ is called a trace-algebra if it has an involution $x \to x^*$, a trace (a positive linear functional) tr defined on it, and has the following properties (here x, y are arbitrary members of B):

(1) $\text{tr}(xy) = \text{tr}(yx)$.

(2) $\text{tr}(x^*x) = n(x^*x)$.

(3) $n(x^*) = n(x)$.

(4) $|\text{tr}(x)| \leq n(x)$.

(5) if $x \neq 0$ then $x^*x \neq 0$.

We also make the standard assumption "$n(xy) \leq n(x) \cdot n(y)$, $x, y \in B$," about the continuity of multiplication.

Let B be a trace-algebra. Let $(\ ,\ )$ be the scalar product on B defined in terms of the trace, $(x, y) = \text{tr}(y^*x) = \text{tr}(xy^*)$ $(x, y \in B)$. Then B is a pre-Hilbert space. Let $\|\ \|$ be the corresponding norm and let A be the completion of B with respect to this norm.

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LEMMA 2. *"$\|x\| \leqslant n(x)$" holds for each $x \in B$.*

PROOF. Direct verification:

$$\|x\|^2 = \operatorname{tr}(x^*x) = n(x^*x) \leqslant n(x^*)n(x) = n(x)^2.$$

LEMMA 3. *Multiplication of B is continuous with respect to the Hilbert space norm, $\|xy\| \leqslant \|x\| \cdot \|y\|$, for all $x, y \in B$.*

PROOF. We verify directly:

$$\|xy\|^2 = \operatorname{tr}(y^*x^*xy) = \operatorname{tr}(yy^*x^*x) = (x^*x, yy^*) \leqslant \|x^*x\| \cdot \|yy^*\|$$
$$\leqslant n(x^*x) \cdot n(yy^*) = \operatorname{tr}(x^*x)\operatorname{tr}(yy^*) = \|x\|^2\operatorname{tr}(y^*y) = \|x\|^2 \cdot \|y\|^2.$$

THEOREM 1. *The completion A of the trace-algebra B is a proper H^*-algebra.*

PROOF. The fact that A is an H^*-algebra is easily verified. If $x, y, z \in B$ then

$$(xy, z) = \operatorname{tr}(z^*xy) = (y, x^*z) = \operatorname{tr}(yz^*x) = \operatorname{tr}((zy^*)^*x) = (x, zy^*).$$

The involution is extendable, as an isometry, to entire A; it has the same property.

Let us show that A is proper. Let T be the trivial ideal [1, p. 371] of A. Then $A = T \oplus T^p$ and the orthogonal complement T^p of T is a proper H^*-algebra. If $T \neq 0$ then there exists some member a of B such that $a \notin T^p$. Write $a = x + y$ with $x \in T, y \in T^p$. Then $x \neq 0$ and $\|a\|^2 = \|x\|^2 + \|y\|^2$. On the other hand we have $a^*a = (x + y)^*(x + y) = y^*y$ since $TA = AT = 0$. This simply means that $\|y\|^2 = \operatorname{tr}(y^*y) = \operatorname{tr}(a^*a) = \|a\|^2$, and this is a contradiction; A is proper.

We shall refer to the algebra A above as the H^*-algebra associated with the (trace-algebra) B.

THEOREM 2 (Characterization of a trace-algebra associated with an H^*-algebra). *Let B be an abstract trace-algebra whose norm $n(\ )$ satisfies the following condition for each $a \in B$.*

$$n(a) = \operatorname{lub}\Big\{|\operatorname{tr}(xa)|: \operatorname*{lub}_{n(y^*y) \leqslant 1} \operatorname{tr}(y^*x^*xy) \leqslant 1\Big\}. \qquad (*)$$

Then there exists a proper H^*-algebra A such that $\tau(A) = B$.

PROOF. Let A be the H^*-algebra associated with B. We only need to show that $\tau(A) = B$. Let $x, y \in B$ and $a = xy$. Then $n(a) = \tau(a)$ because of Lemma 1 above. If $x, y \in A \sim B$ then there are sequences x_n, y_n of members of B such that $\|x_n - x\| \to 0$ and $\|y - y_n\| \to 0$. Then it is easy to check that $\{x_n y_n\}$ is a Cauchy sequence in the norm $n(\ )$:

$$n(x_n y_n - x_m y_m) \leqslant n(x_n(y_n - y_m)) + n((x_n - x_m)y_m)$$
$$= \tau(x_n(y_n - y_m)) + \tau((x_n - x_m)y_m)$$
$$\leqslant \|x_n\| \cdot \|y_n - y_m\| + \|x_n - x_m\| \cdot \|y_m\| \to 0.$$

(Here we used Corollary 4 on p. 99 of [5].) Let a' be its limit, $\lim_n n(a' - x_n y_n) = 0$. It follows that $\|a' - x_n y_n\| \to 0$. But $\|xy - x_n y_n\| \to 0$, hence $a' = xy$, and so $\tau(A) \subset B$.

Conversely let $a \in B$ and consider the functional f_a: $S \to \mathrm{tr}(Sa)$ on the space $C(A)$ of right centralizers of A [6, p. 101]. For each $x \in A$ consider the centralizer Lx: $y \to xy$ acting on A. Then $\|Lx\| = \mathrm{lub}\{|\mathrm{tr}(y^*x^*xy)|: y \in B, n(y^*y) \leq 1\}$, since B is dense in A, and so $\|f_a\| = \mathrm{lub}\{|\mathrm{tr}(xa)|: x \in B, \|Lx\| \leq 1\} = n(a)$ is finite. (The last equality follows from the condition (*) in the statement of the theorem.) Invoking Theorem 1 of [6] we conclude that $a \in \tau(A)$. Thus $B \subset \tau(A)$.

COROLLARY (Characterization of the trace-class (τc) of operators on a Hilbert space). *For each simple trace-algebra B satisfying condition* (*) *of Theorem* 2 *above there exists a Hilbert space H such that B is isomorphic and isometric to the trace-class* (τc) [4, *p*. 36] *of operators acting on H.*

PROOF. It is easy to see that the algebra A associated with B is simple. It follows then from the second structure theorem for H^*-algebras (Theorem 4.3 on p. 380 of [1]) that A can be identified with the algebra (σc) [4, p. 29] of Hilbert-Schmidt operators acting on the Hilbert space $H = L^2(\Gamma)$, where $\Gamma = \{e_\alpha\}$ is a maximal family of primitive doubly orthogonal selfadjoint idempotents of A. Then B could be identified with the trace-class (τc) of operators acting on H.

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