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ON CLOSED STARSHAPED SETS AND BAIRE CATEGORY

GERALD BEER

ABSTRACT. Let C be a closed set of second category in a normed linear space, and let C^* be the subset of C each point of which sees all points of C except a set of first category. If C^* is nonempty, then C^* is a closed convex set. Moreover, $C = K \cup P$ where K is a closed starshaped set with convex kernel C^* and P is a set of first category.

Let C be a set in a linear space. If $\{x, y\} \subset C$ and the line segment joining x to y lies in C, then we say that x *sees* y *via* C. The set of points that x sees via C is called the *star* of x (relative to C). If the star of some point x in C is the entire set, then we say that C is *starshaped* with respect to x, and we call the set of points that see each point of C the *convex kernel* of C. If the linear space is equipped with a norm, then we can talk about points in C that see essentially all of C in a topological sense. It is the purpose of this article to characterize closed sets in a normed linear space that have such points.

Before proceeding, we set forth some notation and additional terminology. If x is a vector in a normed linear space L, then $B_\lambda(x)$ will denote the open ball with radius λ and center x. If $y \neq x$, then seg$[x, y]$ will symbolize the closed line segment joining the two points. If $C \subset L$ and x is in C, then $S(x)$ will denote the star of x. A subset C of a topological space is called *nowhere dense* if its closure has empty interior. It is called a *set of first category* if and only if it can be expressed as a countable union of nowhere dense sets; otherwise, it is called a *set of second category*. If the relative complement of a subset K of C is of first category, then K is called a *residual subset* of C. If C is a subset of a normed linear space, then C^* will denote those points in C whose star is a residual subset of C. Thus, if C is of second category, then C^* is that subset of C whose points see all but an insignificant set of points in a topological sense.

LEMMA 1. *Let L be a normed linear space. If $x \in C \subset L$ and the star $S(x)$ of x relative to C is a set of second category, then $S(x) \cap B_\lambda(x)$ is a set of second category for each positive λ.*

PROOF. Without loss of generality we can assume that $\lambda < 1$. We can represent $S(x)$ as follows:

$$S(x) = \bigcup_{n=1}^{\infty} B_n(x) \cap S(x).$$

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Since the sets of first category form a σ-ideal of sets, there exists n such that $E = B_n(x) \cap S(x)$ is of second category. Hence, the image of E under the homeomorphism $g(z) = x + (\lambda/n)(z - x)$ is a subset of $B_\lambda(x)$ of second category. Morever, since $\lambda/n < 1$, we have $g(E) \subset S(x)$. Hence, $S(x) \cap B_\lambda(x)$ is of second category.

COROLLARY. *Let C be a closed set in a normed linear space. If C^* is nonempty, then the star of each point in C^* includes $\{x \in C$: $S(x)$ is of second category$\}$.*

PROOF. Let $y \in C^*$ be arbitrary. If $S(x)$ is of second category, Lemma 1 implies that y sees points in each neighborhood of x. Thus, x is in $S(y)$ because $S(y)$ is a closed set.

The previous corollary serves to trivially characterize those closed starshaped sets C in a normed linear space for which the star of each point is of second category: C^*, the set of points in C whose star is a residual subset of C, is nonempty. Notice also that since C^* will be the convex kernel of such a set C, it must necessarily be a closed convex set [4]. But what if some points of C see only a set of first category via C? If C^* is nonempty, is the set still closed and convex, and how is C related to a starshaped set? It is the purpose of this article to answer these questions.

Our major tools will be variants of two basic theorems in category theory [3]: a polar form of the Kuratowski-Ulam Theorem and a general form of the Banach Category Theorem.

THEOREM 1. *Let L be a normed linear space and let E be a subset of first category. For each x of norm one, let $E_x = \{\alpha: \alpha > 0$ and $\alpha x \in E\}$. Then $\{x: E_x$ is of second category in $R\}$ is a subset of first category in the relative topology for $\Delta = \{x: \|x\| = 1\}$.*

PROOF. Since the assignment $E \to E_x$ preserves set operations, it suffices to show that if E is a nowhere dense closed set, then E_x is nowhere dense for all x except a set of first category in Δ. Denote the dense open set $L - E$ by G, and let $\{V_n: n \in Z^+\}$ be a countable collection of intervals that serve as a base for the usual topology on $(0,\infty)$. For each positive integer n define G_n as follows:

$$G_n = \{x: \alpha x \in G \text{ for some } \alpha \text{ in } V_n\} \cap \Delta.$$

We first show that G_n is open in the relative topology. To see this, fix x_0 in G_n and choose α_0 in V_n such that $\alpha_0 x_0 \in G$. By the continuity of the map $f(x) = \alpha_0 x$ at $x = x_0$, there exists a neighborhood W of x_0 such that $f(W) \subset G$. Thus, $W \cap \Delta \subset G_n$, and G_n is open.

We next verify that G_n is dense in the relative topology. Let U be an open set in L that meets Δ. We must show that $G_n \cap U \neq \varnothing$. Define a subset T of L as follows:

$$T = \{\alpha x: x \in U \cap \Delta \text{ and } \alpha \in V_n\}.$$

If we could show that T contains an open set, then $T \cap G \neq \varnothing$ whence $U \cap G_n \neq \varnothing$. We shall show that T is actually open. To this end fix x_0 in $U \cap \Delta$ and α_0 in V_n. Suppose for each positive integer n we have $B_{1/n}(\alpha_0 x_0) \cap \tilde{T} \neq \varnothing$. For each n

choose x_n in $B_{1/n}(\alpha_0 x_0) \cap \tilde{T}$. Since $x_n/\|x_n\|$ is in U eventually, it follows that $\|x_n\|$ is not in V_n for all n sufficiently large. Since this contradicts $\lim_{n\to\infty}\|x_n\| = \|\alpha_0 x_0\| = \alpha_0$, there exists a neighborhood of $\alpha_0 x_0$ contained in T. From the above remarks it follows that G_n is dense.

The proof is completed by observing that for each x in $\bigcap_{n=1}^{\infty} G_n$ the set E_x is a nowhere dense subset of $(0, \infty)$.

Although an uncountable union of sets of first category might occasionally produce a set of first category, it is surprising that under certain nonexotic conditions such an occurence will not be accidental. The conditions presented below appear in [2] and represent a refinement of a theorem of Banach [1].

BANACH CATEGORY THEOREM. *Let Y be a subspace of a topological space X and let $\{V_\lambda\colon \lambda \in \Lambda\}$ be a collection of relatively open sets each of which is of first category in X (though not necessarily in Y). Then $\bigcup_{\lambda\in\Lambda} V_\lambda$ is of first category in X.*

We are now ready to expose the structure of closed sets in a normed linear space having points that see a residual subset.

THEOREM 2. *Let C be a closed set of second category in a normed linear space L. Suppose that $C^* = \{x \in C\colon C - S(x)$ is of first catergory$\}$ is nonempty. Then*

(a) *C^* is a closed convex set,*

(b) *$C = K \cup P$ where K is a closed starshaped set with convex kernel C^* and P is of first category.*

PROOF. (a) We first show that C^* is a closed set. Let $\{x_n\}$ be a sequence in C^* convergent to a point x (which must be in C). Clearly,

$$C - S(x) \subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} [C - S(x_k)].$$

Since $C - S(x_k)$ is of first category for each k and the sets of first category form a σ-ideal, $C - S(x)$ is of first category. Hence, x is in C^*. To establish the convexity of C^*, let z and y be distinct members of C^*. Since z sees a set of second category via C, the corollary to Lemma 1 implies that y sees z via C. We have shown that $\text{seg}[z, y] \subset C$, but it remains to show that $\text{seg}[z, y] \subset C^*$.

Without loss of generality we can assume that $z = 0$. By Theorem 1 the set A of points x of norm one such that $\{\alpha\colon \alpha > 0$ and $\alpha x \in S(0) - S(y)\}$ is of second category in the line is of first category in the boundary of the unit ball. It easily follows that $A' = \{\alpha x\colon \alpha > 0$ and $x \in A\}$ is of first category in L. Now if w is in $S(0) - A'$, then w cannot be in $S(y)^{\sim}$ because $\{\alpha\colon \alpha w/\|w\| \in S(0) - S(y)\}$ cannot contain an interval, a set of second category in the line. Hence, if w is in $S(0) - A'$, then w is in $S(y)$. We have shown that y sees every ray or line segment in $S(0)$ with one endpoint 0 in its entirety except for a subset of $S(0)$ of first category. Hence, each point of $\text{seg}[0,y]$ has the same property. Since $S(0) - A'$ is a residual subset of C, the convexity of C^* is established.

(b) Since the relative complement of the star of each point in C^* is relatively open and is of first category in L, the Banach Category Theorem implies that $K = \bigcap_{x\in C^*} S(x)$ is a closed residual subset of C. We now claim that each point of K sees each point of C^* via K, not merely via C. Fix p in K and y in C^*. If z is any other point of C^*, then since seg$[y, z] \subset C^*$, it follows that p sees seg$[y, z]$ via C. Thus, the convex hull of $\{p, z, y\}$ is contained in C so that seg$[p, y] \subset S(z)$. Since z was arbitrary, seg$[p, y] \subset K$ and K is starshaped with respect to C^*. By definition C^* is contained in the convex kernel of K, and since K is a residual subset of C, the reverse inclusion holds, too. We finally remark that Lemma 1 implies that the star of each point of $C - K$ is a set of first category. Hence, C is the union of a closed starshaped set plus a set of insignificant size each point of which sees only a set of insignificant size.

We close by noting that the above decomposition theorem as a consequence of the Banach Category Theorem rests on the axiom of choice. However, a constructive proof can be obtained if L is required to be separable: in this case K can be taken to be the points mutually visible from a countable dense subset of C^*, or alternatively, one could provide a constructive proof of the Banach Category Theorem for second countable spaces.

References

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Department of Mathematics, California State University, Los Angeles, California 90032