

## ON CLOSED STARSHAPED SETS AND BAIRE CATEGORY

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**ABSTRACT.** Let  $C$  be a closed set of second category in a normed linear space, and let  $C^*$  be the subset of  $C$  each point of which sees all points of  $C$  except a set of first category. If  $C^*$  is nonempty, then  $C^*$  is a closed convex set. Moreover,  $C = K \cup P$  where  $K$  is a closed starshaped set with convex kernel  $C^*$  and  $P$  is a set of first category.

Let  $C$  be a set in a linear space. If  $\{x, y\} \subset C$  and the line segment joining  $x$  to  $y$  lies in  $C$ , then we say that  $x$  sees  $y$  via  $C$ . The set of points that  $x$  sees via  $C$  is called the *star* of  $x$  (relative to  $C$ ). If the star of some point  $x$  in  $C$  is the entire set, then we say that  $C$  is *starshaped* with respect to  $x$ , and we call the set of points that see each point of  $C$  the *convex kernel* of  $C$ . If the linear space is equipped with a norm, then we can talk about points in  $C$  that see essentially all of  $C$  in a topological sense. It is the purpose of this article to characterize closed sets in a normed linear space that have such points.

Before proceeding, we set forth some notation and additional terminology. If  $x$  is a vector in a normed linear space  $L$ , then  $B_\lambda(x)$  will denote the open ball with radius  $\lambda$  and center  $x$ . If  $y \neq x$ , then  $\text{seg}[x, y]$  will symbolize the closed line segment joining the two points. If  $C \subset L$  and  $x$  is in  $C$ , then  $S(x)$  will denote the star of  $x$ . A subset  $C$  of a topological space is called *nowhere dense* if its closure has empty interior. It is called a *set of first category* if and only if it can be expressed as a countable union of nowhere dense sets; otherwise, it is called a *set of second category*. If the relative complement of a subset  $K$  of  $C$  is of first category, then  $K$  is called a *residual subset* of  $C$ . If  $C$  is a subset of a normed linear space, then  $C^*$  will denote those points in  $C$  whose star is a residual subset of  $C$ . Thus, if  $C$  is of second category, then  $C^*$  is that subset of  $C$  whose points see all but an insignificant set of points in a topological sense.

**LEMMA 1.** *Let  $L$  be a normed linear space. If  $x \in C \subset L$  and the star  $S(x)$  of  $x$  relative to  $C$  is a set of second category, then  $S(x) \cap B_\lambda(x)$  is a set of second category for each positive  $\lambda$ .*

**PROOF.** Without loss of generality we can assume that  $\lambda < 1$ . We can represent  $S(x)$  as follows:

$$S(x) = \bigcup_{n=1}^{\infty} B_n(x) \cap S(x).$$

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Since the sets of first category form a  $\sigma$ -ideal of sets, there exists  $n$  such that  $E = B_n(x) \cap S(x)$  is of second category. Hence, the image of  $E$  under the homeomorphism  $g(z) = x + (\lambda/n)(z - x)$  is a subset of  $B_\lambda(x)$  of second category. Moreover, since  $\lambda/n < 1$ , we have  $g(E) \subset S(x)$ . Hence,  $S(x) \cap B_\lambda(x)$  is of second category.

**COROLLARY.** *Let  $C$  be a closed set in a normed linear space. If  $C^*$  is nonempty, then the star of each point in  $C^*$  includes  $\{x \in C : S(x) \text{ is of second category}\}$ .*

**PROOF.** Let  $y \in C^*$  be arbitrary. If  $S(x)$  is of second category, Lemma 1 implies that  $y$  sees points in each neighborhood of  $x$ . Thus,  $x$  is in  $S(y)$  because  $S(y)$  is a closed set.

The previous corollary serves to trivially characterize those closed starshaped sets  $C$  in a normed linear space for which the star of each point is of second category:  $C^*$ , the set of points in  $C$  whose star is a residual subset of  $C$ , is nonempty. Notice also that since  $C^*$  will be the convex kernel of such a set  $C$ , it must necessarily be a closed convex set [4]. But what if some points of  $C$  see only a set of first category via  $C$ ? If  $C^*$  is nonempty, is the set still closed and convex, and how is  $C$  related to a starshaped set? It is the purpose of this article to answer these questions.

Our major tools will be variants of two basic theorems in category theory [3]: a polar form of the Kuratowski-Ulam Theorem and a general form of the Banach Category Theorem.

**THEOREM 1.** *Let  $L$  be a normed linear space and let  $E$  be a subset of first category. For each  $x$  of norm one, let  $E_x = \{\alpha : \alpha > 0 \text{ and } \alpha x \in E\}$ . Then  $\{x : E_x \text{ is of second category in } R\}$  is a subset of first category in the relative topology for  $\Delta = \{x : \|x\| = 1\}$ .*

**PROOF.** Since the assignment  $E \rightarrow E_x$  preserves set operations, it suffices to show that if  $E$  is a nowhere dense closed set, then  $E_x$  is nowhere dense for all  $x$  except a set of first category in  $\Delta$ . Denote the dense open set  $L - E$  by  $G$ , and let  $\{V_n : n \in \mathbb{Z}^+\}$  be a countable collection of intervals that serve as a base for the usual topology on  $(0, \infty)$ . For each positive integer  $n$  define  $G_n$  as follows:

$$G_n = \{x : \alpha x \in G \text{ for some } \alpha \text{ in } V_n\} \cap \Delta.$$

We first show that  $G_n$  is open in the relative topology. To see this, fix  $x_0$  in  $G_n$  and choose  $\alpha_0$  in  $V_n$  such that  $\alpha_0 x_0 \in G$ . By the continuity of the map  $f(x) = \alpha_0 x$  at  $x = x_0$ , there exists a neighborhood  $W$  of  $x_0$  such that  $f(W) \subset G$ . Thus,  $W \cap \Delta \subset G_n$ , and  $G_n$  is open.

We next verify that  $G_n$  is dense in the relative topology. Let  $U$  be an open set in  $L$  that meets  $\Delta$ . We must show that  $G_n \cap U \neq \emptyset$ . Define a subset  $T$  of  $L$  as follows:

$$T = \{\alpha x : x \in U \cap \Delta \text{ and } \alpha \in V_n\}.$$

If we could show that  $T$  contains an open set, then  $T \cap G \neq \emptyset$  whence  $U \cap G_n \neq \emptyset$ . We shall show that  $T$  is actually open. To this end fix  $x_0$  in  $U \cap \Delta$  and  $\alpha_0$  in  $V_n$ . Suppose for each positive integer  $n$  we have  $B_{1/n}(\alpha_0 x_0) \cap \tilde{T} \neq \emptyset$ . For each  $n$

choose  $x_n$  in  $B_{1/n}(\alpha_0 x_0) \cap \tilde{T}$ . Since  $x_n/\|x_n\|$  is in  $U$  eventually, it follows that  $\|x_n\|$  is not in  $V_n$  for all  $n$  sufficiently large. Since this contradicts  $\lim_{n \rightarrow \infty} \|x_n\| = \|\alpha_0 x_0\| = \alpha_0$ , there exists a neighborhood of  $\alpha_0 x_0$  contained in  $T$ . From the above remarks it follows that  $G_n$  is dense.

The proof is completed by observing that for each  $x$  in  $\bigcap_{n=1}^{\infty} G_n$  the set  $E_x$  is a nowhere dense subset of  $(0, \infty)$ .

Although an uncountable union of sets of first category might occasionally produce a set of first category, it is surprising that under certain nonexotic conditions such an occurrence will not be accidental. The conditions presented below appear in [2] and represent a refinement of a theorem of Banach [1].

**BANACH CATEGORY THEOREM.** *Let  $Y$  be a subspace of a topological space  $X$  and let  $\{V_\lambda: \lambda \in \Lambda\}$  be a collection of relatively open sets each of which is of first category in  $X$  (though not necessarily in  $Y$ ). Then  $\bigcup_{\lambda \in \Lambda} V_\lambda$  is of first category in  $X$ .*

We are now ready to expose the structure of closed sets in a normed linear space having points that see a residual subset.

**THEOREM 2.** *Let  $C$  be a closed set of second category in a normed linear space  $L$ . Suppose that  $C^* = \{x \in C: C - S(x) \text{ is of first category}\}$  is nonempty. Then*

- (a)  $C^*$  is a closed convex set,
- (b)  $C = K \cup P$  where  $K$  is a closed starshaped set with convex kernel  $C^*$  and  $P$  is of first category.

**PROOF.** (a) We first show that  $C^*$  is a closed set. Let  $\{x_n\}$  be a sequence in  $C^*$  convergent to a point  $x$  (which must be in  $C$ ). Clearly,

$$C - S(x) \subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} [C - S(x_k)].$$

Since  $C - S(x_k)$  is of first category for each  $k$  and the sets of first category form a  $\sigma$ -ideal,  $C - S(x)$  is of first category. Hence,  $x$  is in  $C^*$ . To establish the convexity of  $C^*$ , let  $z$  and  $y$  be distinct members of  $C^*$ . Since  $z$  sees a set of second category via  $C$ , the corollary to Lemma 1 implies that  $y$  sees  $z$  via  $C$ . We have shown that  $\text{seg}[z, y] \subset C$ , but it remains to show that  $\text{seg}[z, y] \subset C^*$ .

Without loss of generality we can assume that  $z = 0$ . By Theorem 1 the set  $A$  of points  $x$  of norm one such that  $\{\alpha: \alpha > 0 \text{ and } \alpha x \in S(0) - S(y)\}$  is of second category in the line is of first category in the boundary of the unit ball. It easily follows that  $A' = \{\alpha x: \alpha > 0 \text{ and } x \in A\}$  is of first category in  $L$ . Now if  $w$  is in  $S(0) - A'$ , then  $w$  cannot be in  $S(y)^\sim$  because  $\{\alpha: \alpha w/\|w\| \in S(0) - S(y)\}$  cannot contain an interval, a set of second category in the line. Hence, if  $w$  is in  $S(0) - A'$ , then  $w$  is in  $S(y)$ . We have shown that  $y$  sees every ray or line segment in  $S(0)$  with one endpoint 0 in its entirety except for a subset of  $S(0)$  of first category. Hence, each point of  $\text{seg}[0, y]$  has the same property. Since  $S(0) - A'$  is a residual subset of  $C$ , the convexity of  $C^*$  is established.

(b) Since the relative complement of the star of each point in  $C^*$  is relatively open and is of first category in  $L$ , the Banach Category Theorem implies that  $K = \bigcap_{x \in C^*} S(x)$  is a closed residual subset of  $C$ . We now claim that each point

of  $K$  sees each point of  $C^*$  via  $K$ , not merely via  $C$ . Fix  $p$  in  $K$  and  $y$  in  $C^*$ . If  $z$  is any other point of  $C^*$ , then since  $\text{seg}[y, z] \subset C^*$ , it follows that  $p$  sees  $\text{seg}[y, z]$  via  $C$ . Thus, the convex hull of  $\{p, z, y\}$  is contained in  $C$  so that  $\text{seg}[p, y] \subset S(z)$ . Since  $z$  was arbitrary,  $\text{seg}[p, y] \subset K$  and  $K$  is starshaped with respect to  $C^*$ . By definition  $C^*$  is contained in the convex kernel of  $K$ , and since  $K$  is a residual subset of  $C$ , the reverse inclusion holds, too. We finally remark that Lemma 1 implies that the star of each point of  $C - K$  is a set of first category. Hence,  $C$  is the union of a closed starshaped set plus a set of insignificant size each point of which sees only a set of insignificant size.

We close by noting that the above decomposition theorem as a consequence of the Banach Category Theorem rests on the axiom of choice. However, a constructive proof can be obtained if  $L$  is required to be separable: in this case  $K$  can be taken to be the points mutually visible from a countable dense subset of  $C^*$ , or alternatively, one could provide a constructive proof of the Banach Category Theorem for second countable spaces.

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