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THE CEVA PROPERTY CHARACTERIZES REAL, STRICTLY CONVEX BANACH SPACES

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ABSTRACT. If a, b, c are distinct collinear points of a metric space, then

$$(a, b, c) = \begin{cases} ab/bc & \text{if } b \text{ is between } a \text{ and } c, \\ -(ab/bc) & \text{otherwise.} \end{cases}$$

A metric space satisfies the Ceva Property provided for each triple of noncollinear points p, q, r, if s, t, u are points distinct from p, q, r on the metric lines $L(p, q)$, $L(q, r)$, and $L(r, p)$, respectively, then the metric lines $L(p, t)$, $L(q, u)$, and $L(r, s)$ have a common point if and only if $(p, s, q)(q, t, r)(r, u, p) = 1$, and $pq/ps \neq pu/pr$. In the euclidean plane, the requirement that $pq/ps \neq pu/pr$ forces the lines $L(r, s)$ and $L(q, u)$ to have a common point. Thus the case of parallel lines is avoided and the Ceva Property is meaningful in an arbitrary metric space. The main result of the paper is that a complete, convex, externally convex, metric space is a strictly convex Banach space over the reals if and only if it satisfies the Ceva Property.

1. Introduction. The theorem of Ceva and its converse are of importance in euclidean plane geometry. These theorems state that three lines drawn from the vertices of a triangle to points on the opposite sides are concurrent if and only if the product of the signed ratios in which the sides are divided by the three points is 1.

The purpose of this paper is to show the theorem of Ceva and its converse, with appropriate modifications, characterize real, strictly convex Banach spaces among the class of complete, convex, externally convex, metric spaces.

We first note that three parallel lines in the euclidean plane are said to be concurrent; and it is in this context that Ceva's theorem and its converse are valid. Since parallelism has little or no meaning in a metric space, we must rule out this exceptional case. Moreover, directed distances have no meaning in a metric space; but in order to obtain the characterization we must consider signed ratios. For example, if p, q, r are noncollinear points in the euclidean plane, and if q is the midpoint of p and s, t is the midpoint of q and r, and if u is between p and r and $ru/up = 1/2$, then

$$(ps/sq)(qt/tr)(ru/up) = 1$$

but the lines $L(p, t)$, $L(r, s)$ and $L(q, u)$ are not concurrent.

We make the following definition for signed ratios.

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DEFINITION 1.1. If a, b, c are distinct collinear points of a metric space, then

$$(a, b, c) = \begin{cases} ab/bc & \text{if } b \text{ is between } a \text{ and } c, \\ -(ab/bc) & \text{otherwise.} \end{cases}$$

It is now easy to state the appropriate modifications of Ceva's theorem and its converse.

CEVA PROPERTY. If p, q, r are noncollinear points of a metric space and if s, t, u are points distinct from p, q, r on metric lines $L(p, q)$, $L(q, r)$, and $L(r, p)$, respectively, then the metric lines $L(p, t)$, $L(q, u)$, and $L(r, s)$ are concurrent, (have a common point), if and only if

$$(p, s, q)(q, t, r)(r, u, p) = 1,$$

and $pq/ps \neq pu/pr$.

In the euclidean plane, the requirement that $pq/ps \neq pu/pr$ forces the lines $L(r, s)$ and $L(q, u)$ to have a common point. Thus the case of parallel lines is avoided and the Ceva Property is meaningful in an arbitrary metric space.

Since algebraic lines are the only metric lines in a strictly convex Banach space, a point z is on a line $L(x, y)$ if and only if a real number λ exists such that $z = \lambda x + (1 - \lambda)y$. Furthermore, for such a point, we have $xz/zy = |1 - \lambda|/|\lambda|$. Since each pair of 2-dimensional Banach spaces over the reals are topologically isomorphic, and topological isomorphisms preserve algebraic lines and points of intersection of algebraic lines, it is easily seen that the Ceva Property of the euclidean plane is preserved under a topological isomorphism between the euclidean plane and a real, strictly convex, 2-dimensional Banach space. Thus all real, strictly convex Banach spaces have the Ceva Property.

Throughout the remainder of this paper, M will denote a complete, convex, externally convex, metric space which has the Ceva Property. For a detailed study of the notions of convex, externally convex and the notation used throughout, the reader is referred to [2].

2. Consequences of the Ceva Property. We now derive a sequence of lemmas which follow fairly quickly from the Ceva Property. These lemmas give geometric structure of the space and lead to the characterization, which we postpone until the next section.

LEMMA 2.1. *Two distinct points of M lie on exactly one (metric) line.*

PROOF. Since M is complete, convex, and externally convex, each two distinct points of M lie on at least one line. If lines are not unique, then distinct points p, q, r, s can be found such that q is a midpoint of p and r and of p and s. Let t be a point between r and s such that $rt/ts = 1/4$. Then p, r, s are noncollinear and

$$(p, q, r)(r, t, s)(s, q, p) \neq 1,$$

but $L(p, t)$, $L(r, q)$ and $L(s, q)$ are concurrent at p. This contradicts the fact that M has the Ceva Property.

LEMMA 2.2. *If p, q, r are noncollinear points of M and if s and u are points on the (metric) segments $S(p, q)$ and $S(p, r)$, respectively, such that $ps/pq = pu/pr$, then the line through p and t, the midpoint of q and r, bisects the segment joining s and u.*

PROOF. Let m denote the midpoint of s and u. Since $ps/pq = pu/pr$, $ps/sq = pu/ur$, and the triples p, s, q and p, u, r satisfy the same betweenness relation. Consequently,

$$(p, s, q)(q, t, r)(r, u, p) = 1,$$

and by the Ceva Property, $L(p, t)$, $L(q, u)$ and $L(r, s)$ have a common point, say v. By the same reasoning,

$$(p, q, s)(s, m, u)(u, r, p) = 1,$$

and $L(s, r)$, $L(u, q)$, and $L(p, m)$ are concurrent. Since lines are unique by Lemma 2.1, $L(p, v)$ coincides with $L(p, m)$ and $L(p, t)$.

LEMMA 2.3. *If p, q, r are noncollinear points of M and if s, u are any points between p and q and p and r, respectively, then $L(r, s)$ and $L(q, u)$ have a common point.*

PROOF. Let t be the point on the segment joining q and r such that $qt/tr = (sq/ps)(up/ru)$. Then

$$(p, s, q)(q, t, r)(r, u, p) = (ps/sq)(qt/tr)(ru/up) = 1,$$

and by the Ceva Property $L(p, t)$, $L(q, u)$ and $L(r, s)$ have a common point.

LEMMA 2.4. *Let p, q, r be noncollinear points. If s, t, u are points on the segments $S(p, q)$, $S(q, r)$ and $S(r, p)$, respectively, such that*

$$(p, s, q)(q, t, r)(r, u, p) = 1,$$

then the point, v, common to $L(p, t)$, $L(q, u)$ and $L(r, s)$ is between p and t, between q and u, and between r and s.

PROOF. Since v, q, r are noncollinear points and $L(q, s)$, $L(v, t)$, and $L(r, u)$ have the point p in common,

$$(v, u, q)(q, t, r)(r, s, v) = 1.$$

By hypothesis, t is between q and r, and thus $(q, t, r) > 0$. Consequently, $(v, u, q)(r, s, v) > 0$.

By way of contradiction, suppose $(v, u, q) > 0$ and $(r, s, v) > 0$. Then

$$rs + sv = rv < vu + ur = vu + pr - pu \tag{1}$$

and

$$vu + uq = qv < qs + sv = sv + pq - ps. \tag{2}$$

Adding the respective extreme sides of the inequalities of (1) and (2), cancelling common terms, and adding $pu + ps$ to both sides of the resulting inequality, we obtain

$$rs + uq + pu + ps < pr + pq,$$

which contradicts the triangle inequality. Therefore, $(v, u, q) < 0$ and $(r, s, v) < 0$. Thus u is not between v and q and s is not between r and v. It follows that v is between q and u or q is between u and v and v is between r and s or r is between s and v.

Suppose q is between u and v and r is between s and v. Then

$$uq + qv = uv < ur + rv = rv + pr - pu \tag{3}$$

and

$$sr + rv = sv < qv + qs = qv + pq - ps; \tag{4}$$

as above we obtain

$$uq + sr + pu + ps < pr + pq,$$

contrary to the triangle inequality.

We now have v is between q and u or v is between r and s. We need to show both relations are valid.

Without loss of generality, suppose v is between q and u and v is not between r and s. Since $(r, s, v) < 0$, s is not between r and v, so r is between s and v.

Since p, s, u are noncollinear points, we may pick a point x on $L(s, u)$ such that

$$(p, q, s)(s, x, u)(u, r, p) = 1.$$

Since $(p, q, s) < 0$ and $(u, r, p) < 0$ by hypothesis, $(s, x, u) > 0$. Moreover, by the Ceva Property, $L(p, x)$, $L(r, s)$ and $L(u, q)$ are concurrent. Since lines are unique and $L(r, s)$ and $L(q, u)$ contain the point v, v is on $L(p, x)$.

Since s, u, v are noncollinear and $L(s, q)$, $L(u, r)$, and $L(v, x)$ all contain p, by the Ceva Property

$$(s, r, v)(v, q, u)(u, x, s) = 1.$$

This is impossible, for $(s, r, v) > 0$ and $(v, q, u) < 0$ by assumption and $(s, x, u) > 0$ by the above. We conclude that v is between q and u and v is between r and s. In a similar manner it is seen that v is between p and t.

COROLLARY 2.1. *If p, q, r are noncollinear points of M and if s, u are any points between p and q and p and r, respectively, then the segments $S(r, s)$ and $S(q, u)$ have a common interior point.*

COROLLARY 2.2. *If p, q, r are noncollinear points of M and if s, t, u are any points between p and q, q and r, and r and p, respectively, then the segments $S(u, s)$ and $S(p, t)$ have a common interior point.*

PROOF. By Corollary 2.1, the segments $S(p, t)$ and $S(q, u)$ have a common point v. A second application of Corollary 2.1 to the noncollinear points p, u, q with s between p and q and v between q and u yields the result.

COROLLARY 2.3. *If p, q, r are noncollinear points and if s and u are between p and q and p and r, respectively, then the line $L(s, u)$ contains no point between q and r.*

PROOF. Suppose the contrary. Then $L(s, u)$ contains a point t that is between q and r. By Corollary 2.2, the segments $S(r, s)$ and $S(u, t)$ have a common interior point v. But now $L(r, s)$ and $L(u, t)$ have the distinct points s and v in common, contrary to the fact that two distinct points lie on a unique line.

COROLLARY 2.4. *If p, q, r are noncollinear points and if s and u are points between p and q and p and r, respectively, then the line $L(q, r)$ contains no point between s and u.*

PROOF. Suppose $L(q, r)$ contains a point v between s and u. By Corollary 2.3, v is not between q and r. No generality is lost if we assume q is between v and r. Since p, v, and r are noncollinear points and q is between v and r and u is between p and r, by Corollary 2.1, the segments $S(v, u)$ and $S(p, q)$ have a common interior point t. But now the lines $L(p, q)$ and $L(u, s)$ have the distinct points s and t in common, contrary to Lemma 2.1.

LEMMA 2.5. *Let p, q, r be noncollinear points of M. If u is between p and r and if v is between q and u, then $L(r, v)$ contains a point w between p and q.*

PROOF. By Corollary 2.1, for each point x on the segment $S(p, q)$, $S(r, x)$ and $S(q, u)$ have exactly one common point, this defines a function f from $S(p, q)$ into $S(q, u)$. Suppose $\{x_n\}$ is a sequence of points of $S(p, q)$ with $\lim x_n = x_0$. Then $\{f(x_n)\}$ is a sequence of the compact set $S(q, u)$ and consequently, it contains a convergent subsequence, which we may assume is the original sequence. If $\lim f(x_n) = y_0$, then since $rf(x_n) + x_nf(x_n) = rx_n$, by continuity of the metric, $ry_0 + y_0x_0 = rx_0$. Thus, y_0 is on $S(r, x_0)$ and y_0 is on $S(q, u)$, and by the definition of $f, y_0 = f(x_0)$; that is, f is continuous on $S(p, q)$. Now f maps the connected set $S(p, q)$ onto a connected subset of $S(q, u)$. But $f(q) = q$ and $f(p) = u$ and f is onto. This completes the proof, since f is $1 - 1$.

LEMMA 2.6. *If p, q, r are noncollinear points and if s, t, u are points on the lines $L(p, q)$, $L(q, r)$, and $L(r, p)$, respectively, such that p is the midpoint of q and s, p is the midpoint of r and u, and t is the midpoint of q and r, then $L(p, t)$ bisects the segment joining u and s.*

PROOF. Let x and y be points on $L(p, q)$ and $L(p, r)$, respectively, such that q is the midpoint of p and x and r is the midpoint of p and y. If m is the midpoint of x and y, then p, x, y are noncollinear points and

$$(p, s, x)(x, m, y)(y, u, p) = 1.$$

By the Ceva Property $L(x, u)$, $L(y, s)$ and $L(p, m)$ all contain a point, say z. Since m is the midpoint of x and y and q, r are the respective midpoints of p and x and p and y, by Lemma 2.2, t is on $L(p, m)$. Now z, x, y are noncollinear points and $L(y, u)$, $L(x,s)$ and $L(z, m)$ all contain p, so by the Ceva Property,

$$(z, u, x)(x, m, y)(y, s, z) = 1.$$

By Lemma 2.2, $L(z, m)$ contains the midpoint of u and s, which completes the proof, since $L(z, m) = L(p, t)$.

LEMMA 2.7. *Let p, q, r be noncollinear points and let s, t, u be points on the lines $L(p, q)$, $L(q, r)$ and $L(r, p)$, respectively, such that p is the midpoint of q and s, p is the midpoint of r and u, and t is the midpoint of q and r. If a, v, w are the respective midpoints of u and s, u and q, and r and s, then p is the midpoint of a and t and p is the midpoint of v and w.*

PROOF. It follows from Lemma 2.6 that a, p, t are collinear points and v, p, w are collinear points.

Choose sequences $\{s_n\}$ and $\{u_n\}$ on the segments $S(p, s)$ and $S(p, u)$, respectively, such that $ps_n/ps = pu_n/pu$ and $\lim s_n = s$ and $\lim u_n = u$. It follows from the Ceva Property that the lines $L(q, u_n)$, $L(r, s_n)$, and $L(p, t)$ contain a common point t_n and by Lemma 2.2 $L(p, t)$ bisects $S(u_n, s_n)$. Since s_n is between p and s and w is between r and s, the segments $S(w, p)$ and $S(r, s_n)$ have a common point w_n by Corollary 2.1. Since w_n is on the compact set $S(p, w)$, $\{w_n\}$ contains a subsequence which converges to a point on $S(p, w)$, say w_0. No generality is lost if we assume the original sequence converges to w_0. Now $s_n w_n + w_n r = s_n r$ and by continuity of the metric, $sw_0 + w_0 r = sr$ and since w_0 is also on the segment $S(p, w)$, we have $w_0 = w$. For each n let r_n be the midpoint of s_n and r. Then

$$s_n r_n + r_n w_n = s_n w_n$$

or

$$s_n w_n + w_n r_n = s_n r_n.$$

In any event, since $\lim s_n w_n = sw$ and $\lim s_n r_n = sw$, $\lim w_n r_n = 0$. Now $r_n w < r_n w_n + w_n w$ and since $\lim r_n w_n = 0$ and $\lim w_n w = 0$, it follows that $\lim r_n w = 0$; that is $\lim r_n = w$.

Similarly if v_n is the point of intersection of the segments $S(q, u_n)$ and $S(p, v)$, then $\lim v_n = v$ and if q_n is the midpoint of u_n and q then $\lim q_n = v$. Now by Lemma 2.2, $L(p, t)$ bisects the segment $S(r_n, q_n)$ in a point p_n. An argument similar to that given above shows $\lim p_n = p$. It now follows from the continuity of the metric that p is the midpoint of v and w. In the same manner, it is seen that p is the midpoint of a and t.

3. The characterization. At this point it is convenient to introduce the Young Postulate and show the Ceva Property implies it. This will complete the characterization, for Andalafte and Blumenthal [1] have shown a complete, convex, externally convex, metric space is a Banach space if and only if it satisfies the Young Postulate.

THE YOUNG POSTULATE. If p, q, and r are points of a metric space, and if q' and r' are the midpoints of p and q, and of p and r, respectively, then $q'r' = qr/2$.

THEOREM 3.1. *The space M satisfies the Young Postulate.*

PROOF. Since each pair of distinct points of M lies on a unique line (Lemma 2.1), it is easily seen that collinear points satisfy the Young Postulate. Thus, we need only show that if p, q, r are noncollinear points of M and if q', r' are the respective midpoints of p and q and p and r, then $q'r' = qr/2$.

Let m be the point on the line $L(q, r')$ such that r' is the midpoint of q and m and let $\{m'_n\}$ be a sequence of distinct points on the segment $S(m, r)$ such that $\lim m'_n = m$. By Corollary 1, the segments $S(r, r')$ and $S(q, m'_n)$ have a common point r_n, $n = 1, 2, 3, \ldots$. The sequence $\{r_n\}$, a subset of the compact set $S(r, r')$, contains a convergent subsequence which we may assume to be the original sequence. Since $qr_n + r_n m'_n = qm'_n$ and $\lim qm'_n = qm$, it follows that $\lim r_n = r_0$ is between q and m and since r_0 is on the segment $S(r, r')$ we conclude $\lim r_n = r'$. Moreover, p, q, r_n are noncollinear points. For each n, let m_n be the point on $L(q, r_n)$ such that

$$(p, q', q)(q, m_n, r_n)(r_n, r, p) = 1. \tag{5}$$

By the Ceva Property, $L(p, m_n)$, $L(r_n, q')$, and $L(q, r)$ contain a common point, say c_n. Since q' is the midpoint of p and q, $(p, q', q) = 1$ and $(q, m_n, r_n) = 1/(r_n, r, p)$. Since r_n is between r and p, $(r_n, r, p) < 0$ and $rr_n < pr$. Consequently, $(q, m_n, r_n) < 0$ and m_n is not between q and r_n. Moreover, $qm_n > m_n r_n$ and thus r_n is between q and m_n.

For each n, let t_n be the point on the line $L(r, m_n)$ such that

$$(m_n, q, r_n)(r_n, p, r)(r, t_n, m_n) = 1. \tag{6}$$

By the Ceva Property, the lines $L(p, m_n)$, $L(r_n, t_n)$ and $L(q, r)$ have a common point, say d_n. But from the above, $L(p, m_n)$ and $L(q, r)$ contain the point c_n, hence $d_n = c_n$. Moreover, $(r_n, p, r) < 0$ and $(m_n, q, r_n) < 0$ and thus $(r, t_n, m_n) > 0$ and consequently t_n is between r and m_n, $n = 1, 2, 3, \ldots$.

Now m_n, r_n, c_n are noncollinear points and the lines $L(m_n, t_n)$, $L(r_n, p)$, and $L(c_n, q)$ all contain the point r. Consequently, by the Ceva Property,

$$(m_n, q, r_n)(r_n, t_n, c_n)(c_n, p, m_n) = 1. \tag{7}$$

Since $(p, q', q) = 1$ and $\lim r_n = r'$ implies $\lim(r_n, r, p) = -1/2$, it follows from (5) that $\lim(q, m_n, r_n) = -2$. The fact that r_n is between q and m_n now implies that $\lim(m_n, q, r_n) = -2$ and thus from (7) we have

$$\lim(r_n, t_n, c_n)(c_n, p, m_n) = -\tfrac{1}{2}. \tag{8}$$

Since p, r_n, c_n are noncollinear points and the lines $L(p, q')$, $L(r_n, m_n)$ and $L(c_n, r)$ all contain q, the Ceva Property yields

$$(p, r, r_n)(r_n, q', c_n)(c_n, m_n, p) = 1 \tag{9}$$

and since $\lim(p, r, r_n) = -2$,

$$\lim(r_n, q', c_n)(c_n, m_n, p) = -\tfrac{1}{2}. \tag{10}$$

Since p, q, c_n are noncollinear points and the lines $L(p, r)$, $L(q, m_n)$ and $L(c_n, q')$ all contain r_n,

$$(p, q', q)(q, r, c_n)(c_n, m_n, p) = 1 \tag{11}$$

by the Ceva Property.

Thus $(q, r, c_n)(c_n, m_n, p) = 1$ (since $(p, q', q) = 1$) and (q, r, c_n) and (c_n, m_n, p) are both positive, or both are negative. By Corollaries 2.3 and 2.4, c_n is not between m_n and p and c_n is not between q and r. Consequently, $(q, r, c_n) > 0$ and $(c_n, m_n, p) > 0$ imply

(i) r is between q and c_n and m_n is between c_n and p, while $(q, r, c_n) < 0$ and $(c_n, m_n, p) < 0$ imply

(ii) p is between c_n and m_n and q is between c_n and r.

Since t_n is between m_n and r and q' is between p and q, (i) and Corollary 2.2 imply

(iii) t_n is between q' and c_n,

while (ii) and Corollary 2.2 imply

(iv) q' is between t_n and c_n.

Moreover, (i) and Corollary 2.1 imply t_n is between c_n and r_n and r_n is between q' and c_n and t_n is between q' and c_n. Corollary 2.1 and (ii) imply r_n is between c_n and t_n and q' is between c_n and r_n.

Since $(c_n, p, m_n)(c_n, m_n, p) < 0$ and $(r_n, t_n, c_n)(r_n, q', c_n) < 0$, the result of dividing the respective sides of (8) by the respective sides of (10) is

$$\lim(r_nt_n/r_nq')(c_np/c_nm_n)(q'c_n/t_nc_n) = 1. \tag{12}$$

ASSERTION. $\lim r_nt_n/r_nq' = 1$.

From (5), since $(p, q', q) = 1$ and $\lim(r_n, r, p) = -1/2$, we have $\lim qm_n/m_nr_n = 2$ and thus $\lim(qm_n - 2m_nr_n) = 0$. But $qm_n - 2m_nr_n = qr_n - r_nm_n$ and $\lim qr_n = qr'$ and hence $\lim r_nm_n = qr'$. Now since $qm_n = qr_n + r_nm_n$ we have $\lim qm_n = 2qr' = qm$. Further, since $\lim m_n' = m$, $\lim qm_n' = qm$. Since $qm = qm_n' \pm m_nm_n'$, $\lim m_nm_n' = 0$. Now $0 \leqslant mm_n \leqslant m_nm_n' + m_n'm$, so $\lim m_n = m$. Let t_0 denote the midpoint of r and m. If m_n is between q and m_n', an application of Corollary 2.1 shows that $L(q, t_0)$ contains a point between r and m_n. If m_n' is between q and m_n, then by Lemma 2.5, $L(q, t_0)$ contains a point between r and m_n. Let s_n denote the point on $L(q, t_0)$ that is between r and m_n, $n = 1, 2, 3, \ldots$. Since $\lim m_n = m$, the sequence $\{s_n\}$ is a bounded subset of the finitely compact set $L(q, t_0)$. Thus, $\{s_n\}$ contains a convergent subsequence which we may assume to be the original sequence. If $\lim s_n = s_0$, then s_0 is on $L(q, t_0)$ and since $rs_n + s_nm_n = rm_n$, by continuity of the metric s_0 is on $L(r, m)$. Since $L(r, m)$ and $L(q, t_0)$ have t_0 in common, $s_0 = t_0$. Since $\lim(m_n, q, r_n) = -2$ and $\lim(r_n, p, r) = 1/2$, from (6) $\lim(r, t_n, m_n) = 1$. Thus $0 = \lim(rt_n - t_nm_n) = \lim(rm_n - 2t_nm_n)$ and since $\lim rm_n = rm$, $\lim t_nm_n = rt_0$ and consequently $\lim rt_0 = rt_n$. Now $rs_n = rt_n \pm t_ns_n$ and since $\lim rs_n = rt_0$ and $\lim rt_n = rt_0$, $\lim t_ns_n = 0$. Now $0 \leqslant t_nt_0 \leqslant s_nt_n + s_nt_0$ and $\lim t_n = t_0$. By Lemma 2.7, $r't_0/r'q' = 1$ and by continuity of the metric $\lim(r_nt_n/r_nq') = 1$.

Returning to (12), we see that $\lim(c_np/c_nm_n)(q'c_n/t_nc_n) = 1$. Using (i), (ii), (iii), and (iv), we have

$$\lim[(c_nm_n + m_np)/c_nm_n][(q't_n + t_nc_n)/t_nc_n] = 1 \tag{13}$$

or

$$\lim[(c_nm_n - m_np)/c_nm_n][(c_nt_n - q't_n)/t_nc_n] = 1, \tag{14}$$

and consequently $\lim m_np/c_nm_n = \lim q't_n/t_nc_n = 0$. Since $\lim m_np = mp$ and $\lim q't_n = q't_0$, $\lim c_nm_n = \lim t_nc_n = \infty$ and since $t_nc_n - t_nr < rc_n$, $\lim rc_n = \infty$. Also $q'c_n = t_nc_n \pm q't_n$ and $\lim q'c_n = \infty$. Now $r_nc_n/q'c_n = (q'c_n \pm q'r_n)/q'c_n$ and $\lim r_nc_n/q'c_n = 1$.

Dividing the respective side of (9) by the respective sides of (11) and simplifying the result we have

$$(r_nq'/qr)(pr/r_nr)(rc_n/q'c_n) = 1.$$

Since $\lim pr/r_n r = 2$ and $\lim rc_n/q'c_n = 1$, $1/2 = \lim r_n q'/qr_n = r'q'/qr$; that is $r'q' = qr/2$.

THEOREM 3.2. *A complete, convex, externally convex, metric space is a real, strictly convex Banach space if and only if it has the Ceva Property.*

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