INTERPOLATION FAILS FOR
THE SOUSLIN-KLEENE CLOSURE OF
THE OPEN SET QUANTIFIER LOGIC

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ABSTRACT. In this paper we show that the Souslin-Kleene closure of the open set quantifier logic fails to have interpolation. We also show that the notion of a $T_0$-topological space is not definable in this logic. This gives a more natural proof that it is strictly weaker than the interior operator logic.

The questions of whether the Souslin-Kleene closure of the open set quantifier logic is the interior operator logic or even has interpolation come naturally from the work of the author in [3], [4], and [5]. We begin by giving the formal definitions and then the three results which settle these questions.

DEFINITION. The Souslin-Kleene closure, $\Delta(\mathcal{L}^*)$, of a logic $\mathcal{L}^*$ is the logic formed by adding the complementary pseudo-elementary classes to the elementary classes.

That is, $\Omega$ and $\Omega^c$ are $PC_{\mathcal{L}}(L)$-classes if and only if they are $EC_{\Delta(\mathcal{L}^*)}(L)$-classes. See [1] for further background.

DEFINITION. Take a structure $\mathfrak{A}$ and $q \subseteq \mathcal{P}(A)$ and form $(\mathfrak{A}, q)$. If $q$ is a topology on $A$ then $(\mathfrak{A}, q)$ is called topological.

DEFINITION. The open set quantifier logics $\mathcal{L}(Q)$ and $\mathcal{L}(Q_n)_{n \in \omega}$ are formed by adding quantifiers $Qx$ and $Q\bar{x}$, $n \in \omega$, to first order logic where the interpretations of $Q\varphi(x)$ and $Q\bar{x} \varphi(\bar{x})$, respectively, are that the sets defined by $\varphi(x)$ and $\varphi(\bar{x})$ are open in the topology and the $n$th product topology. For further background see [3] and [4].

We are now ready to state and prove the main theorems of this paper.

THEOREM 1. For each $(\mathfrak{A}, q)$ where $\mathfrak{A}$ is an $L$-structure there is an $L^* \supseteq L$ and an extension $\mathfrak{A}$ of $\mathfrak{A}$ to $L^*$ such that if $(\mathfrak{A}, q) \equiv_{\mathcal{L}(Q^*)}(\mathfrak{A}^*, q)$ then

$$(\mathfrak{A} \upharpoonright L, r) \equiv_{\Delta(\mathcal{L}(Q))}(\mathfrak{A}, q).$$

PROOF. This result is a straightforward application of the definition of $\Delta(\mathcal{L}(Q))$.

COUNTEREXAMPLE 2. The counterexample to interpolation for $\Delta(\mathcal{L}(Q_n))$ is the same as the one for $\mathcal{L}(Q)$ as presented in [3] and [4].

We will assume that interpolation holds and derive a contradiction. Let $L_1 = \{B(x), C(x), R(x)\}$ and $L_2 = \{B(x), C(x), P(x)\}$. We define $q(P)$ to be

$$\neg QxB(x) \land \forall y(B(y) \leftrightarrow C(y) \lor R(y)) \land QxR(x)$$

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and $\psi(P)$ to be
\[ \forall x (C(x) \rightarrow P(x)) \rightarrow \neg Qx(P(x) \land B(x)). \]

One easily sees that $\models \varphi(R) \rightarrow \psi(P)$. Take $A = \mathbb{N}$, i.e. the set of natural numbers
\[ B^\mathbb{N} = \{2n | n \in \mathbb{N}\}, \quad C^\mathbb{N} = \{4n | n \in \mathbb{N}\}. \]

Define $(\mathbb{N}, q)$ to be $\langle A, B^\mathbb{N}, C^\mathbb{N}, (\emptyset, \mathbb{N}) \rangle$.

Now since we have assumed that $\Delta(E(Q))$ has interpolation there is a $\theta \in \Delta(E(Q))$ such that $\models \varphi(R) \rightarrow \theta$ and $\models \neg \psi(P) \rightarrow \neg \theta$.

Without loss of generality we assume that $(\mathbb{N}, q)$ models $\theta$ since the argument in the alternate case is entirely analogous.

Now expand $\mathbb{A}$ to an $L^\#$-structure $\mathbb{A}^\#$ as in Theorem 1. We then will expand $q$ to a $q^\#$ and define a $\mathbb{P}^\#$ such that $(\mathbb{A}^\#, q) \prec_{E(Q)} (\mathbb{A}^\#, q^\#)$ and $(\mathbb{A}, \mathbb{P}^\#, q^\#) \models \neg \psi(p)$. This implies that $(\mathbb{A}, q^\#) \equiv_{\Delta(E(Q))} (\mathbb{A}, q)$ and $(\mathbb{A}, q^\#) \models \neg \theta$ which is a contradiction.

Let $\psi(x), i \in \omega$, enumerate the $L^\#_*(Q)$ definable nonopen sets of $(\mathbb{A}^\#, q)$. We proceed by induction. For 0 we pick an $x$ and $y$ such that
\[ y \in [\psi_0(x)]^{(\mathbb{A}^\#, q)} \]
and
\[ x \in B^{\mathbb{A}^\#} - ([\psi_0(x)]^{(\mathbb{A}^\#, q)}) \]
if possible, otherwise
\[ x \in A - ([\psi_0(x)]^{(\mathbb{A}^\#, q)}). \]

Assume we have picked the sequences $y_0, \ldots, y_n$ and $x_0, \ldots, x_n$. We will now choose $x_{n+1}$ and $y_{n+1}$ as follows. Choose, if possible, $x \in B^{\mathbb{A}^\#} - [\psi_{n+1}(x)]^{(\mathbb{A}^\#, q)}$ such that $x \neq y_i$ for $0 < i < n$. Otherwise pick $y$ such that
\[ y \in [\psi_{n+1}(x)]^{(\mathbb{A}^\#, q)} \cap B^{\mathbb{A}^\#} \]
and
\[ y \neq x_i \quad \text{for } 0 < i < n. \]

This is possible since otherwise $[\psi_{n+1}(x)]^{(\mathbb{A}^\#, q)} \cap B^{\mathbb{A}^\#}$ would be finite which would imply that $B^{\mathbb{A}^\#}$ is finite which is false.

Let $\mathbb{P}^{\mathbb{A}^\#}$ be $(C^{\mathbb{A}^\#} \cup \{x_i \}_{i \in \omega}) \cap B^{\mathbb{A}^\#}$ and let $q^\#$ be the topology generated by $q \cup \{\mathbb{P}^{\mathbb{A}^\#}\}$. We claim that $(\mathbb{A}^\#, q) \prec_{E(Q)} (\mathbb{A}^\#, q^\#)$ and that $(\mathbb{A}^\#, \mathbb{P}^{\mathbb{A}^\#}, q^\#) \models \neg \psi(p)$. The second clause is straightforward. We prove the first by induction on the number of occurrences of $Qx$.

If $(\mathbb{A}^\#, q) \models Qx\varphi(x)$ then $(\mathbb{A}^\#, q^\#) \models Qx\varphi(x)$ since $q \subseteq q^\#$, thus assume $(\mathbb{A}^\#, q^\#) \models Qx\varphi(x)$ and $(\mathbb{A}^\#, q) \models \neg Qx\varphi(x)$ and derive a contradiction. Thus $[\varphi(x)]^{(\mathbb{A}^\#, q)} = \mathbb{P}^{\mathbb{A}^\#}$. But there is a $k$ such that $[\varphi(x)]^{(\mathbb{A}^\#, q)} = [\psi_k(x)]^{(\mathbb{A}^\#, q)}$ so by the definition of $\mathbb{P}^{\mathbb{A}^\#}$ either
\[ x_k \in \mathbb{P}^{\mathbb{A}^\#} - [\psi_k(x)]^{(\mathbb{A}^\#, q)}. \]
or

\[ y_k \in [\psi_k(x)](\mathbb{R}^+,q) - P^\mathbb{R}\].

Hence a contradiction.

**Remark.** The analogous result for \( \Delta(\mathcal{E}(Q^n)_{n \in \omega}) \) can be proved by the same method. Also this result shows that the interior operator logics \( \mathcal{E}(I) \) and \( \mathcal{E}(I^n)_{n \in \omega} \) as defined in \([3]\) and \([5]\) strictly contain \( \Delta(\mathcal{E}(Q)) \) and \( \Delta(\mathcal{E}(Q^n)_{n \in \omega}) \), respectively, since they both have interpolation by \([5]\).

By \([2]\) we know that because \( \Delta(\mathcal{E}(Q)) \) and \( \Delta(\mathcal{E}(Q^n)_{n \in \omega}) \) do not have interpolation they do not have a Beth definability theorem.

However this result of strict containment can be improved by giving a more natural counterexample in the topological sense.

**Definition.** A topological space is called \( T_0 \) (Minkowski), if and only if for each \( x \neq y \) there is an open set containing one but not the other.

We can equivalently define a \( T_0 \)-space as a space where unequal points have unequal closures. See \([6]\).

The class of \( T_0 \)-spaces is the class of models of the \( \mathcal{E}(I) \) sentence

\[ \forall x \forall y (x \neq y \rightarrow (Iy(y \neq x) \lor Ix(x \neq y))). \]

However we will now prove that the class of \( T_0 \) models is not a basic elementary class of \( \Delta(\mathcal{E}(Q)) \).

Take \( \mathcal{A} \) to be \( \mathcal{A} = \{ f | f : \{0, 1\} \rightarrow \mathbb{N} \} \) and \( L = \emptyset \).

Define a pseudometric by \( d(x,y) = |x(0) - y(0)| \). Then the topology that \( d \) generates, call it \( q \), is generated by the closures of points and every open set is infinite. \( (\mathcal{A}, q) \) also is not a \( T_0 \)-space since the closure of a point, which is infinite, is the closure of any point in it.

Now we will construct the counterexample using the following theorem.

**Theorem 3.** There is a topology \( q^* \) such that \( (\mathcal{A}, q^*) \) is a \( T_0 \)-topology and \( (\mathcal{A}, q) \cong_{\Delta(\mathcal{E}(Q))}(\mathcal{A}, q^*) \).

**Proof.** To show this result we expand \( \mathcal{A} \) and \( L \) to \( \mathcal{A}^* \) and \( L^* \) as in Theorem 1 (taking pains to add functions to the language to pick out noninterior points from definable nonopen sets as in \([3]\)).

Given a pair \( a, b \) we will define a topology \( q_{a,b} \) such that \( (\mathcal{A}^*, q) \cong_{\Delta(\mathcal{E}(Q))}(\mathcal{A}, q^*) \), \( a \) and \( b \) have unequal closures, and \( q_{a,b} \) is generated by the closures of points and every open set is infinite. This is the same topological property of \( (\mathcal{A}, q) \) which we use.

We then iterate this construction through all distinct pairs and take the union (see \([3]\)) which will be \( T_0 \) and satisfy the conclusion to the theorem.

Define \( x_{-1} = a \) and \( y_{-1} = b \). Take \( h \) to be a bijection from \( \mathbb{N} \) into \( \mathbb{N} \times \mathbb{N} \times 2 \). Let \( \psi_i(x) \), \( i \in \omega \), enumerate the \( L^*_\mathbb{R}(Q) \) definable nonopen sets and let \( \emptyset_i \), \( i \in \omega \), enumerate the closures of points, which is a basis for \( q \).

Assume we have defined \( x_{-1}, \ldots, x_{n-1}, y_{-1}, \ldots, y_{n-1} \). We now will define \( x_n \) and \( y_n \).
Assume \((h(n))_2 = 0\). Pick an \(x\) such that \(x \neq y_i\), \(-1 < i < n - 1\), and,
\[x \in \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}\text{ and } \Theta(h(n))_0 \subseteq \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}\]
or
\[x \in \Theta(h(n))_0 - \left( \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)} \cup \{ y_{i-1} \}_{0 < i < n} \right),\]
and set \(y_n = y_{n-1}\) and \(x_n = x\).

Otherwise pick a \(y \in \Theta(h(n))_0 \cap \left( \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)} - \text{Int}(\left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}) \right)\)
and \(x \in \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}\)
and set \(y_n = y\), \(x_n = x\), where \(\text{Int}(X)\) is the interior of the set.

If \((h(n))_2 = 1\) then switch \(x\) and \(y\).

This definition is possible because if
\[\Theta(h(n))_0 \supseteq \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)},\]
and
\[\Theta(h(n))_0 - \left( \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)} \cup \{ y_{i-1} \}_{0 < i < n} \right) = \emptyset,\]
then \(\Theta(h(n))_0 - \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}\) is nonempty and finite. Take a \(y' \in \Theta(h(n))_0 - \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}\). Then \(\text{Cl}(y') \cap \Theta(h(n))_0\) is open, infinite and contains \(y'\). Hence there is a \(y\) such that
\[y \in \Theta(h(n))_0 \cap \left( \left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)} - \text{Int}(\left[ \psi(h(n))_1(x) \right]^{(\mathbb{R}^+, q)}) \right)\].

Let \(\Theta = \{ x_{i-1} \}_{i \in \omega}\) and \(q_{(a,b)}\) be the topology generated by \(q\), \(\Theta\), and \(N - \Theta\). \(\Theta\)
and \(N - \Theta\) are infinite because both of the sets \(\{ m | (h(n))_2 = 0 \}\) and \(\{ n | (h(n))_2 = 1 \}\) are infinite.

Now \(a \in \Theta\) and \(b \not\in \Theta\) and each set is infinite so all we need to show is that
\((\mathbb{R}^+, q) \leq a \cup q_{(a,b)}\).

We show this by induction on the number of occurrences of \(Qx\). Since \(q \leq q_{(a,b)}\) we need only to show one direction. So assume that \((\mathbb{R}^+, q_{(a,b)}) \vdash Qx\varphi(x)\) and
\((\mathbb{R}^+, q) \vdash \neg Qx\varphi(x)\) and derive a contradiction. Hence
\[\left[ \varphi(x) \right]^{(\mathbb{R}^+, q_{(a,b)})} = \left[ \varphi(x) \right]^{(\mathbb{R}^+, q)} = (\Theta_a \cap \Theta) \cup (\Theta_b \cap \Theta^c).\]

Either \(\Theta_a\) or \(\Theta_b\) is not a subset of \(\left[ \varphi(x) \right]^{(\mathbb{R}^+, q)}\) since otherwise \(\Theta_a \cup \Theta_b = \left[ \varphi(x) \right]^{(\mathbb{R}^+, q)}\). So assume \(\Theta_a \subseteq \left[ \varphi(x) \right]^{(\mathbb{R}^+, q)}\) since the other case follows by symmetry.

There are \(k, l\) such that \(\Theta_k \subseteq \Theta_a, \Theta_k\) basic open, \(\left[ \psi_l(x) \right]^{(\mathbb{R}^+, q)} = \left[ \varphi(x) \right]^{(\mathbb{R}^+, q)}\) and \(\Theta_k \subseteq \left[ \psi_l(x) \right]^{(\mathbb{R}^+, q)}\). Take \(h^{-1}(k, l, 0) = m\) and we have
\[\Theta_k - \left( \left[ \psi_l(x) \right]^{(\mathbb{R}^+, q)} \cup \{ y_{i-1} \}_{0 < i < m} \right) = \emptyset,\]
since otherwise \(\Theta_a \cap \Theta \nsubseteq \left[ \varphi(x) \right]^{(\mathbb{R}^+, q)}\). Thus
\[y_m \in (\Theta_k \cap \left[ \psi_l(x) \right]^{(\mathbb{R}^+, q)} - \text{Int}(\left[ \psi_l(x) \right]^{(\mathbb{R}^+, q)}).\]

If \(\Theta_b = \emptyset\) then we are done since \(y_m \not\in \Theta_a \cap \Theta\).
To finish assume $\emptyset_\beta \neq \emptyset$. Then $y_m \in \emptyset_\beta \cap \emptyset^c$. Hence $y_m \in \emptyset_\beta \cap \emptyset_k \subseteq [\psi(x)](\mathbb{R}^n)$, since if

$$\emptyset_\beta \cap \emptyset_k \cap \{y_{i-1}\}_{0 \leq i \leq m} [\psi(x)](\mathbb{R}^n) \neq \emptyset$$

then $\emptyset_\beta \cap \emptyset_k \cap \emptyset^c [\psi(x)](\mathbb{R}^n) \neq \emptyset$ which is a contradiction. But $y_m$ is a noninterior point by definition so we have a contradiction.

We have shown the result and we can prove analogously the same result for $\Delta[\mathcal{L}(Q^n)_{n \in \omega}]$ via the same method.

REFERENCES


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