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INTERPOLATION FAILS FOR THE SOUSLIN-KLEENE CLOSURE OF THE OPEN SET QUANTIFIER LOGIC

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ABSTRACT. In this paper we show that the Souslin-Kleene closure of the open set quantifier logic fails to have interpolation. We also show that the notion of a T_0-topological space is not definable in this logic. This gives a more natural proof that it is strictly weaker than the interior operator logic.

The questions of whether the Souslin-Kleene closure of the open set quantifier logic is the interior operator logic or even has interpolation come naturally from the work of the author in [3], [4], and [5]. We begin by giving the formal definitions and then the three results which settle these questions.

DEFINITION. The *Souslin-Kleene closure*, $\Delta(\mathfrak{L}^*)$, of a logic $\mathfrak{L}^*$ is the logic formed by adding the complementary pseudo-elementary classes to the elementary classes.

That is, Ω and Ω^c are $PC_{\mathfrak{L}^*}(L)$-classes if and only if they are $EC_{\Delta(\mathfrak{L}^*)}(L)$-classes. See [1] for further background.

DEFINITION. Take a structure $\mathfrak{A}$ and $q \subseteq \mathcal{P}(A)$ and form $(\mathfrak{A}, q)$. If q is a topology on A then $(\mathfrak{A}, q)$ is called *topological.*

DEFINITION. The *open set quantifier logics* $\mathfrak{L}(Q)$ and $\mathfrak{L}(Q^n)_{n\in\omega}$ are formed by adding quantifiers Qx and $Q\vec{x}$, $n \in \omega$, to first order logic where the interpretations of $Qx\varphi(x)$ and $Q\vec{x}\varphi(\vec{x})$, respectively, are that the sets defined by $\varphi(x)$ and $\varphi(\vec{x})$ are open in the topology and the nth product topology. For further background see [3] and [4].

We are now ready to state and prove the main theorems of this paper.

THEOREM 1. *For each $(\mathfrak{A}, q)$ where $\mathfrak{A}$ is an L-structure there is an $L^\# \supseteq L$ and an extension $\mathfrak{A}^\#$ of $\mathfrak{A}$ to $L^\#$ such that if $(\mathfrak{B}, r) \equiv_{\mathfrak{L}(Q)} (\mathfrak{A}^\#, q)$ then*

$$(\mathfrak{B} \restriction L, r) \equiv_{\Delta(\mathfrak{L}(Q))} (\mathfrak{A}, q).$$

PROOF. This result is a straightforward application of the definition of $\Delta(\mathfrak{L}(Q))$.

COUNTEREXAMPLE 2. The counterexample to interpolation for $\Delta(\mathfrak{L}(Q^n_{n\in\omega}))$ is the same as the one for $\mathfrak{L}(Q)$ as presented in [3] and [4].

We will assume that interpolation holds and derive a contradiction. Let $L_1 = \{B(x), C(x), R(x)\}$ and $L_2 = \{B(x), C(x), P(x)\}$. We define $\varphi(R)$ to be

$$\neg QxB(x) \wedge \forall y(B(y) \leftrightarrow C(y) \vee R(y)) \wedge QxR(x)$$

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and $\psi(P)$ to be

$$\forall x(C(x) \to P(x)) \to \neg Qx(P(x) \wedge B(x)).$$

One easily sees that $\models \varphi(R) \to \psi(P)$. Take $A = \mathbf{N}$, i.e. the set of natural numbers

$$B^{\mathfrak{A}} = \{2n | n \in \mathbf{N}\}, \qquad C^{\mathfrak{A}} = \{4n | n \in \mathbf{N}\}.$$

Define $(\mathfrak{A}, q)$ to be $\langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, \{\varnothing, \mathbf{N}\}\rangle$.

Now since we have assumed that $\Delta(\mathfrak{L}(Q))$ has interpolation there is a $\theta \in \Delta(\mathfrak{L}(Q))$ such that $\models \varphi(R) \to \theta$ and $\models \neg\psi(P) \to \neg\theta$.

Without loss of generality we assume that $(\mathfrak{A}, q)$ models θ since the argument in the alternate case is entirely analogous.

Now expand $\mathfrak{A}$ to an $L^{\#}$-structure $\mathfrak{A}^{\#}$ as in Theorem 1. We then will expand q to a $q^{\#}$ and define a $P^{\mathfrak{A}^{\#}}$ such that $(\mathfrak{A}^{\#}, q) \prec_{\mathfrak{L}(Q)} (\mathfrak{A}^{\#}, q^{\#})$ and $(\mathfrak{A}, P^{\mathfrak{A}^{\#}}, q^{\#}) \models \neg\psi(p)$. This implies that $(\mathfrak{A}, q^{\#}) \equiv_{\Delta(\mathfrak{L}(Q))} (\mathfrak{A}, q)$ and $(\mathfrak{A}, q^{\#}) \models \neg\theta$ which is a contradiction.

Let $\psi_i(x)$, $i \in \omega$, enumerate the $L_A^{\#}(Q)$ definable nonopen sets of $(\mathfrak{A}^{\#}, q)$. We proceed by induction. For 0 we pick an x and y such that

$$y \in [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}$$

and

$$x \in B^{\mathfrak{A}^{\#}} - [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}$$

if possible, otherwise

$$x \in A - [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}.$$

Assume we have picked the sequences $y_0, \ldots, y_n$ and $x_0, \ldots, x_n$. We will now choose x_{n+1} and y_{n+1} as follows. Choose, if possible, $x \in B^{\mathfrak{A}^{\#}} - [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)}$ such that $x \neq y_i$ for $0 \leqslant i \leqslant n$. Otherwise pick y such that

$$y \in [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)} \cap B^{\mathfrak{A}^{\#}}$$

and

$$y \neq x_i \quad \text{for } 0 \leqslant i \leqslant n.$$

This is possible since otherwise $[\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)} \cap B^{\mathfrak{A}^{\#}}$ and $B^{\mathfrak{A}^{\#}} - [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)}$ would be finite which would imply that $B^{\mathfrak{A}^{\#}}$ is finite which is false.

Let $P^{\mathfrak{A}^{\#}}$ be $(C^{\mathfrak{A}^{\#}} \cup \{x_i\}_{i \in \omega}) \cap B^{\mathfrak{A}^{\#}}$ and let $q^{\#}$ be the topology generated by $q \cup \{P^{\mathfrak{A}^{\#}}\}$. We claim that $(\mathfrak{A}^{\#}, q) \prec_{\mathfrak{L}(Q)} (\mathfrak{A}^{\#}, q^{\#})$ and that $(\mathfrak{A}^{\#}, P^{\mathfrak{A}^{\#}}, q^{\#}) \models \neg\psi(P)$. The second clause is straightforward. We prove the first by induction on the number of occurrences of Qx.

If $(\mathfrak{A}^{\#}, q) \models Qx\varphi(x)$ then $(\mathfrak{A}^{\#}, q^{\#}) \models Qx\varphi(x)$ since $q \subseteq q^{\#}$, thus assume $(\mathfrak{A}^{\#}, q^{\#}) \models Qx\varphi(x)$ and $(\mathfrak{A}^{\#}, q) \models \neg Qx\varphi(x)$ and derive a contradiction. Thus $[\varphi(x)]^{(\mathfrak{A}^{\#}, q)} = P^{\mathfrak{A}^{\#}}$. But there is a k such that $[\varphi(x)]^{(\mathfrak{A}^{\#}, q)} = [\psi_k(x)]^{(\mathfrak{A}^{\#}, q)}$ so by the definition of $P^{\mathfrak{A}^{\#}}$ either

$$x_k \in P^{\mathfrak{A}^{\#}} - [\psi_k(x)]^{(\mathfrak{A}^{\#}, q)}$$

or

$$y_k \in [\psi_k(x)]^{(\mathfrak{A}^*,q)} - P^{\mathfrak{A}^*}.$$

Hence a contradiction.

REMARK. The analogous result for $\Delta(\mathfrak{L}(Q^n)_{n\in\omega})$ can be proved by the same method. Also this result shows that the interior operator logics $\mathfrak{L}(I)$ and $\mathfrak{L}(I^n)_{n\in\omega}$ as defined in [3] and [5] strictly contain $\Delta(\mathfrak{L}(Q))$ and $\Delta(\mathfrak{L}(Q^n)_{n\in\omega})$, respectively, since they both have interpolation by [5].

By [2] we know that because $\Delta(\mathfrak{L}(Q))$ and $\Delta(\mathfrak{L}(Q^n)_{n\in\omega})$ do not have interpolation they do not have a Beth definability theorem.

However this result of strict containment can be improved by giving a more natural counterexample in the topological sense.

DEFINITION. A topological space is called T_0 (*Minkowski*), if and only if for each $x \neq y$ there is an open set containing one but not the other.

We can equivalently define a T_0-space as a space where unequal points have unequal closures. See [6].

The class of T_0-spaces is the class of models of the $\mathfrak{L}(I)$ sentence

$$\forall x \forall y (x \neq y \rightarrow (Iy(y \neq x) \vee Ix(x \neq y))).$$

However we will now prove that the class of T_0 models is not a basic elementary class of $\Delta(\mathfrak{L}(Q))$.

Take $\mathfrak{A}$ to be ${}^2\mathbf{N} = \{f | f: \{0, 1\} \rightarrow \mathbf{N}\}$ and $L = \varnothing$.

Define a pseudometric by $d(x, y) = |x(0) - y(0)|$. Then the topology that d generates, call it q, is generated by the closures of points and every open set is infinite. $(\mathfrak{A}, q)$ also is not a T_0-space since the closure of a point, which is infinite, is the closure of any point in it.

Now we will construct the counterexample using the following theorem.

THEOREM 3. *There is a topology* $q^\#$ *such that* $(\mathfrak{A}, q^\#)$ *is a* T_0*-topology and* $(\mathfrak{A}, q) \equiv_{\Delta(\mathfrak{L}(Q))} (\mathfrak{A}, q^\#)$.

PROOF. To show this result we expand $\mathfrak{A}$ and L to $\mathfrak{A}^\#$ and $L^\#$ as in Theorem 1 (taking pains to add functions to the language to pick out noninterior points from definable nonopen sets as in [3]).

Given a pair a, b we will define a topology $q_{\langle a,b\rangle}$ such that $(\mathfrak{A}^\#, q) \prec_{\mathfrak{L}(Q)} (\mathfrak{A}^\#, q_{\langle a,b\rangle})$, a and b have unequal closures, and $q_{\langle a,b\rangle}$ is generated by the closures of points and every open set is infinite. This is the same topological property of $(\mathfrak{A}, q)$ which we use.

We then iterate this construction through all distinct pairs and take the union (see [3]) which will be T_0 and satisfy the conclusion to the theorem.

Define $x_{-1} = a$ and $y_{-1} = b$. Take h to be a bijection from $\mathbf{N}$ into $\mathbf{N} \times \mathbf{N} \times 2$. Let $\psi_i(x)$, $i \in \omega$, enumerate the $L_A^\#(Q)$ definable nonopen sets and let $\mathcal{O}_i$, $i \in \omega$, enumerate the closures of points, which is a basis for q.

Assume we have defined $x_{-1}, \ldots, x_{n-1}, y_{-1}, \ldots, y_{n-1}$. We now will define x_n and y_n.

Assume $(h(n))_2 = 0$. Pick an x such that $x \neq y_i$, $-1 < i < n-1$, and,

$$x \in [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)} \text{ and } \mathcal{O}_{(h(n))_0} \subseteq [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}$$

or

$$x \in \mathcal{O}_{(h(n))_0} - \left([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)} \cup \{y_{i-1}\}_{0<i<n}\right),$$

and set $y_n = y_{n-1}$ and $x_n = x$.

Otherwise pick a

$$y \in \mathcal{O}_{(h(n))_0} \cap \left([\psi_{(h(n))_1}(x)]\right)^{(\mathfrak{A}^*,q)} - \mathrm{Int}\left([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}\right)$$

and

$$x \in [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}$$

and set $y_n = y$, $x_n = x$, where $\mathrm{Int}(X)$ is the interior of the set.

If $(h(n))_2 = 1$ then switch x and y.

This definition is possible because if

$$\mathcal{O}_{(h(n))_0} \nsubseteq [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)},$$

and

$$\mathcal{O}_{(h(n))_0} - \left([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)} \cup \{y_{i-1}\}_{0<i<n}\right) = \varnothing,$$

then $\mathcal{O}_{(h(n))_0} - [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}$ is nonempty and finite. Take a $y' \in \mathcal{O}_{(h(n))_0} - [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}$. Then $\mathrm{Cl}(y') \cap \mathcal{O}_{(h(n))_0}$ is open, infinite and contains y'. Hence there is a y such that

$$y \in \mathcal{O}_{(h(n))_0} \cap \left([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)} - \mathrm{Int}[\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*,q)}\right).$$

Let $\mathcal{O} = \{x_{i-1}\}_{i\in\omega}$ and $q_{\langle a,b\rangle}$ be the topology generated by q, $\mathcal{O}$, and $\mathbf{N} - \mathcal{O}$. $\mathcal{O}$ and $\mathbf{N} - \mathcal{O}$ are infinite because both of the sets $\{m \mid (h(n))_2 = 0\}$ and $\{n \mid (h(n))_2 = 1\}$ are infinite.

Now $a \in \mathcal{O}$ and $b \notin \mathcal{O}$ and each set is infinite so all we need to show is that $(\mathfrak{A}^\#, q) \prec_{\mathfrak{L}(Q)} (\mathfrak{A}^\#, q_{\langle a,b\rangle})$.

We show this by induction on the number of occurrences of Qx. Since $q \subseteq q_{\langle a,b\rangle}$ we need only to show one direction. So assume that $(\mathfrak{A}^\#, q_{\langle a,b\rangle}) \models Qx\varphi(x)$ and $(\mathfrak{A}^\#, q) \models \neg Qx\varphi(x)$ and derive a contradiction. Hence

$$[\varphi(x)]^{(\mathfrak{A}^*,q_{\langle a,b\rangle})} = [\varphi(x)]^{(\mathfrak{A}^*,q)} = (\mathcal{O}_\alpha \cap \mathcal{O}) \cup (\mathcal{O}_\beta \cap \mathcal{O}^c).$$

Either $\mathcal{O}_\alpha$ or $\mathcal{O}_\beta$ is not a subset of $[\varphi(x)]^{(\mathfrak{A}^*,q)}$ since otherwise $\mathcal{O}_\alpha \cup \mathcal{O}_\beta = [\varphi(x)]^{(\mathfrak{A}^*,q)}$. So assume $\mathcal{O}_\alpha \nsubseteq [\varphi(x)]^{(\mathfrak{A}^*,q)}$ since the other case follows by symmetry.

There are k, l such that $\mathcal{O}_k \subseteq \mathcal{O}_\alpha$, $\mathcal{O}_k$ basic open, $[\psi_l(x)]^{(\mathfrak{A}^*,q)} = [\varphi(x)]^{(\mathfrak{A}^*,q)}$ and $\mathcal{O}_k \nsubseteq [\psi_l(x)]^{(\mathfrak{A}^*,q)}$. Take $h^{-1}(\langle k, l, o\rangle) = m$ and we have

$$\mathcal{O}_k - \left([\psi_l(x)]^{(\mathfrak{A}^*,q)} \cup \{y_{i-1}\}_{0<i<m}\right) = \varnothing$$

since otherwise $\mathcal{O}_\alpha \cap \mathcal{O} \nsubseteq [\varphi(x)]^{(\mathfrak{A}^*,q)}$. Thus

$$y_m \in \left(\mathcal{O}_k \cap [\psi_l(x)]^{(\mathfrak{A}^*,q)} - \mathrm{Int}[\psi_l(x)]^{(\mathfrak{A}^*,q)}\right).$$

If $\mathcal{O}_\beta = \varnothing$ then we are done since $y_m \notin \mathcal{O}_\alpha \cap \mathcal{O}$.

To finish assume $\mathcal{O}_\beta \neq \varnothing$. Then $y_m \in \mathcal{O}_\beta \cap \mathcal{O}^c$. Hence $y_m \in \mathcal{O}_\beta \cap \mathcal{O}_k \subseteq [\psi_l(x)]^{(\mathfrak{A}^*,q)}$, since if

$$\mathcal{O}_\beta \cap \mathcal{O}_k \cap \{y_{i-1}\}_{0<i<m} - [\psi_l(x)]^{(\mathfrak{A}^*,q)} \neq \varnothing$$

then $\mathcal{O}_\beta \cap \mathcal{O}_k \cap \mathcal{O}^c - [\psi_l(x)]^{(\mathfrak{A}^*,q)} \neq \varnothing$ which is a contradiction. But y_m is a noninterior point by definition so we have a contradiction.

We have shown the result and we can prove analogously the same result for $\Delta(\mathfrak{L}(Q^n)_{n\in\omega})$ via the same method.

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