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UNIONS OF CELLS WITH APPLICATIONS TO VISIBILITY

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ABSTRACT. A crumpled n-cell C in E^n is proven to be an n-cell ($n \neq 4$) when it is known to contain two n-cells C_1 and C_2, one of which is flat, such that Bd $C \subset$ (Bd C_1) $\cup$ (Bd C_2). This theorem is applied to show that C is an n-cell if its boundary is the union of two closed sets each of which is seen from some point of Int C. Examples are given to show that flatness of one of C_1 and C_2 is necessary in the first theorem and to show that two is the largest integer for which either theorem is true.

1. Introduction. A crumpled n-cell C in E^n ($n \neq 4$) is known to be an n-cell if it is the union of two n-cells [7]. A modification (see Theorem 2.1) of this result is proven here by showing that C is an n-cell if there are two n-cells C_1 and C_2 in C, one of which is flat in E^n, such that Bd $C \subset$ (Bd C_1) $\cup$ (Bd C_2). Notice that the modification weakens the hypothesis that $C = C_1 \cup C_2$ but imposes flatness on at least one of C_1 and C_2. Example 2.5 shows that this flatness condition is required. Generalizations to more than two n-cells are false as Pixley's examples [8] show.

Theorem 2.1 is applied in §3 to show that the interior of a crumpled n-cell C in E^n is 1-ULC if its boundary is the union of two closed sets each of which is seen from some point of Int C. A set X in Bd C is *seen from a point p* of Int C if, for each $x \in X$, the straight-line segment $[p, x]$ intersects Bd C only at x. The word "visible" in place of "seen" seems more natural, but "visible" has a slightly different meaning in the literature ([3, p. 326], [6, p. 400]). The set X is *visible from a point p* if, for each $x \in X$, the entire ray from p through x intersects Bd C only at x. Thus a visible set is seen but a seen set may not be visible. Example 3.3 shows that being the union of three seen, closed sets is not sufficient evidence to conclude that an $(n - 1)$-sphere in E^n bounds an n-cell.

Earlier work on visible taming conditions was done by Cobb who proved that a 2-sphere S in E^3 is tame if it is locally tame except possibly at points of a closed subset X of S which is visible from a point of Int S. Although Cobb's work has not appeared, Burgess and Cannon [3, p. 327] have provided an outline of a proof. The techniques of their proof are also applicable to "seen" sets and will be used here. Daverman [6, Example 9.4] has an example of a wild $(n - 1)$-sphere S in E^n ($n \geq 4$) that is locally flat modulo a Cantor set visible from Int S. The inflation technique described by Daverman in §9 of [6] has proven particularly useful for

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constructing in E^n ($n \geqslant 4$) wild $(n-1)$-spheres possessing certain geometric properties known to prohibit wildness if $n = 3$. In view of such examples (again, see [**6**, §9]) it may be somewhat surprising that the geometric property of visibility works for $n \geqslant 3$.

The paper concludes with a brief §4 in which two localized versions of the main theorems of §§2 and 3 are given. Though much more cumbersome to state, these local versions are established by the proofs given for their respective global analogues.

2. Flatness of the union of two cells. Theorem 1.2 of [**7**] states that a crumpled n-cube in E^n is an n-cell if it is the union of two n-cells and if $n \neq 4$. Perhaps of importance, other then in its applicaton to visibility theorems in the next section, is the following variation of the result of [**7**]. The proof given for this theorem depends directly on Cannon's Invariance of 1-ULC [**5**, Theorem 4.2] and also establishes a more general local version given in §4. A crumpled n-cube is the closure of a bounded complementary domain of an $(n-1)$-sphere topologically embedded in Euclidean n-space E^n.

THEOREM 2.1. *The interior of a crumpled n-cube C in E^n is* 1-*ULC if there exist two n-cells C_1 and C_2 in C such that* Bd $C \subset (\text{Bd } C_1) \cup (\text{Bd } C_2)$ *and C_2 is flat in E^n. Furthermore C is an n-cell if $n \neq 4$.*

PROOF. Let $S_1 = \text{Bd } C_1$, $S_2 = \text{Bd } C_2$, $S = \text{Bd } C$, $p \in S$, and let $\mathfrak{S}$ be a positive number. Since C_1 is an n-cell there a positive number σ such that each loop in $N(p, \sigma) \cap \text{Int } C_1$ can be shrunk to a point in $N(p, \mathfrak{S}) \cap \text{Int } C_1$. Since S_2 is flat it follows from Theorem 2C.6 of [**5**] (see [**5**, p. 61] for the case $n > 3$) that $E^n - S$ is 1-ULC in $E^n - S \cap S_2$. Consequently there is a positive number δ such that each loop in $N(p, \delta) \cap \text{Int } C$ can be shrunk to a point in $N(p, \sigma) - S \cap S_2$. To show Int C is 1-ULC it suffices to show that each loop in $N(p, \delta) \cap \text{Int } C$ can be shrunk to a point in $N(p, \mathfrak{S}) \cap \text{Int } C$.

Let D denote a 2-cell, and let f be a map taking Bd D into $N(p, \delta) \cap \text{Int } C$. By the choice of δ, f extends to a map, which is still called f, of D into $N(p, \sigma) - S \cap S_2$. The component K of $D - f^{-1}(S \cap f(D))$ containing Bd D contains a finite collection of disjoint simple closed curves whose union separates Bd D from $f^{-1}(S \cap f(D))$. Since $f(D) \cap S_2 = \varnothing$ these curves can be chosen close enough to $f^{-1}(S \cap f(D))$ to insure that their images under f lie in Int C_1. Moreover $f(K) \subset N(p, \sigma)$, so these simple closed curves in K shrink to a point in $N(p, \mathfrak{S}) \cap \text{Int } C_1$. Consequently $f|\text{Bd } D$ can be extended to map all of D into $N(p, \mathfrak{S}) \cap \text{Int } C$.

Bing [**2**] proved that C is a 3-cell if its interior is 1-ULC. A discussion of the analogous result for $n \geqslant 5$ is contained in §5 of [**6**] where references are given.

COROLLARY 2.2. *If C_1 and C_2 are n-cells in S^n ($n \neq 4$) such that C_2 is flat and $S^n - (C_1 \cup C_2)$ has an $(n-1)$-sphere S as its boundary, then S bounds an n-cell containing $C_1 \cup C_2$.*

THEOREM 2.3. *A crumpled cube C in E^3 is a tame* 3-*cell if there exist two tame* 3-*cells in C such that* Bd *C lies in the union of their boundaries.*

PROOF. From Theorem 2.1, C is a 3-cell. Let C_1 and C_2 be two tame 3-cells as hypothesized. It follows from Theorem 4.1 of [**4**] that C_1 and C_2 are *-taming sets since they are tame. Then $C_1 \cup C_2$ is a *-taming set [**4**, Theorem 3.7(1)], and it follows that C is tame.

EXAMPLE 2.4. The failure of Theorem 2.3 for $n > 3$ can be demonstrated by letting C^* be the inflation in S^4 of the Alexander Horned Cube as described by Daverman [**6**, §9], and then taking C as the closure of $S^4 - C^*$.

EXAMPLE 2.5. In Theorem 2.1 and its corollary it is essential that at least one of the two cells C_1 and C_2 be flat. To illustrate this one can take C to be the Alford crumpled cube [**1**], although any crumpled cube whose boundary is locally tame modulo a subset of a simple closed curve J in Bd C will do, provided it is not a 3-cell. Let D_1 and D_2 be disks whose union is Bd C and whose common boundary J contains the wild points of Bd C. Of course each D_i may be assumed locally polyhedral at its interior points, so each Int D_i may be moved slightly toward Int C to obtain a disk D_i' such that $D_i \cap D_i' = J$. Then $D_i \cup D_i'$ bounds a 3-cell C_i in C, and Bd $C \subset$ (Bd C_1) $\cup$ (Bd C_2).

3. Applications to visibility. It is easy to see that an $(n-1)$-sphere S in E^n is flat if S is seen from some point of Int S. The next theorem generalizes this observation to the situation where S is the union of two seen closed sets, however, it turns out that flatness from the seen side (except in E^3 where S is tame from both sides) is all that can be expected. Example 3.3 illustrates this fact. Example 3.4 shows that there is no generalization to the case where S is the union of three or more seen closed sets.

LEMMA 3.1. *If the closed subset X of the $(n-1)$-sphere S in E^n is seen from a point x in* Int S, *then there is a flat n-cell C in $S \cup$* Int S *such that $x \in$* Int C, *$X \subset$* Bd C, *and* Bd C *is visible from x.*

Although the definitions of "seen" and "visible" differ slightly, the proof of Theorem 9.3.2 outlined in [**3**] also establishes Lemma 3.1.

THEOREM 3.2. *The interior of a crumpled n-cell C in E^n is* 1-*ULC if there exist closed sets X_1 and X_2 in* Bd C *and points x_1 and x_2 in* Int C *such that* Bd $C = X_1 \cup X_2$ *and each X_i is seen from x_i $(i = 1, 2)$. Thus for $n \neq 4$, C is an n-cell.*

PROOF. Let $S =$ Bd C and apply Lemma 3.1 twice to obtain two flat n-cells C_1 and C_2 in C such that $S \subset$ (Bd C_1) $\cup$ (Bd C_2). Then Theorem 3.2 follows immediately from Theorem 2.1.

EXAMPLE 3.3. An example of a 4-cell C in E^4 is now described so that Bd C is the union of two closed sets seen from Int C while Ext Bd C is not 1-ULC. Thus one cannot conclude from the hypothesis of Theorem 3.2 that Bd C is nice from both sides (except when $n = 3$ as demonstrated in the proof of Theorem 2.3).

The example C is constructed by inflating the Alexander Horned cube C^1 in $E^3 \times 0$ to a 3-sphere S in E^4 (see [**6**, p. 400]) and defining C as $S \cup$ Ext S. It is particularly easy to find the two seen closed sets if two points x and y are chosen above and below C^1, respectively, before inflating C, because then one may inflate toward these points rather than vertically. The two closed sets X and Y are just the upper and lower hemispheres, respectively, of S since X is seen from x and Y from y. Of course C is not really a crumpled 4-cube but this can be corrected by carefully running a tube out close to $E^3 \times 0$ and then letting it balloon around S so as to interchange the interior and exterior of S. This puts both x and y in Int S, but X and Y need to be enlarged to insure their union is still the entire sphere.

EXAMPLE 3.4. There is a crumpled cube C in E^3 such that its boundary is the union of three closed sets X_1, X_2 and X_3 where each X_i is seen from a point x_i in Int C, yet C is not a 3-cell. In fact the Fox-Artin crumpled cube, whose complement is pictured in [3, p. 270], can be described to satisfy these conditions with the three points x_i collinear.

The main idea for the description is given in §3 of [7]. Let A' be an arc running through the double points of a regular projection P of the Fox-Artin arc F into the xy-plane π (see Figure 4 of [3, p. 270]). There is a homeomorphism h of π onto itself taking A' to an arc A on the x-axis. Now an equivalent embedding of F can be constructed by lifting small "over-arcs" of $h(P)$ into the vertical half-plane $\{(x, y, z) | z \geqslant 0\}$ and making small adjustments in the xy-plane. Thus we may assume F lies on a 3-page book B with two pages in π, binding A, and a third page in the xy-plane (see Figure 1 of [7]).

Let C' be a cube such that B divides C' into three cubes C_1', C_2' and C_3' each with vertical and horizontal sides. By drilling a small tapered hole in C' along F we obtain the Fox-Artin crumpled cube from C'. However before drilling choose three collinear points x_1, x_2 and x_3 in the interiors of C_1', C_2' and C_3', respectively. Now the tube is drilled along F creating grooves in the C_i'. Careful drilling consistent with the location of the x_i will transform C' to the desired crumpled cube C and C_1', C_2' and C_3' to 3-cells C_1, C_2 and C_3 such that Bd C_i is seen from x_i. The closed sets X_i are taken as (Bd C_i) $\cap$ (Bd C).

4. Local versions of the main theorems. The proofs given for Theorems 2.1 and 3.2 actually prove the more general theorems given in this section. These localized versions will be needed in subsequent work.

THEOREM 4.1. *The interior of a crumpled n-cell C in E^n is 1-LC at each point in the interior of $(n-1)$-cell D in* Bd *C if there exist two n-cells C_1 and C_2 in C such that $D \subset$* (Bd C_1) $\cup$ (Bd C_2) *and C_1 is flat in E^n. Furthermore* Bd *C is locally flat at each point of* Int *D if $n \neq 4$.*

THEOREM 4.2. *The interior of a crumpled n-cell C is 1-LC at each point in the interior of an $(n-1)$-cell D in* Bd *C if there exist two closed sets X_1 and X_2 and two points x_1 and x_2 in* Int *C such that $D = X_1 \cup X_2$ and X_i is seen from x_i $(i = 1, 2)$. For $n \neq 4$ it follows that* Bd *C is locally flat at each point of* Int *D.*

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