UNIONS OF CELLS WITH APPLICATIONS TO VISIBILITY

L. D. LOVELAND

Abstract. A crumpled \( n \)-cell \( C \) in \( E^n \) is proven to be an \( n \)-cell \((n \neq 4)\) when it is known to contain two \( n \)-cells \( C_1 \) and \( C_2 \), one of which is flat, such that \( \text{Bd } C \subset (\text{Bd } C_1) \cup (\text{Bd } C_2) \). This theorem is applied to show that \( C \) is an \( n \)-cell if its boundary is the union of two closed sets each of which is seen from some point of \( \text{Int } C \). Examples are given to show that flatness of one of \( C_1 \) and \( C_2 \) is necessary in the first theorem and to show that two is the largest integer for which either theorem is true.

1. Introduction. A crumpled \( n \)-cell \( C \) in \( E^n \) \((n \neq 4)\) is known to be an \( n \)-cell if it is the union of two \( n \)-cells [7]. A modification (see Theorem 2.1) of this result is proven here by showing that \( C \) is an \( n \)-cell if there are two \( n \)-cells \( C_1 \) and \( C_2 \) in \( C \), one of which is flat in \( E^n \), such that \( \text{Bd } C \subset (\text{Bd } C_1) \cup (\text{Bd } C_2) \). Notice that the modification weakens the hypothesis that \( C = C_1 \cup C_2 \) but imposes flatness on at least one of \( C_1 \) and \( C_2 \). Example 2.5 shows that this flatness condition is required. Generalizations to more than two \( n \)-cells are false as Pixley’s examples [8] show.

Theorem 2.1 is applied in §3 to show that the interior of a crumpled \( n \)-cell \( C \) in \( E^n \) is 1-ULC if its boundary is the union of two closed sets each of which is seen from some point of \( \text{Int } C \). A set \( X \) in \( \text{Bd } C \) is \textit{seen} from a point \( p \) of \( \text{Int } C \) if, for each \( x \in X \), the straight-line segment \([p, x] \) intersects \( \text{Bd } C \) only at \( x \). The word “visible” in place of “seen” seems more natural, but “visible” has a slightly different meaning in the literature ([3, p. 326], [6, p. 400]). The set \( X \) is \textit{visible} from a point \( p \) if, for each \( x \in X \), the entire ray from \( p \) through \( x \) intersects \( \text{Bd } C \) only at \( x \). Thus a visible set is seen but a seen set may not be visible. Example 3.3 shows that being the union of three seen, closed sets is not sufficient evidence to conclude that an \((n - 1)\)-sphere in \( E^n \) bounds an \( n \)-cell.

Earlier work on visible taming conditions was done by Cobb who proved that a 2-sphere \( S \) in \( E^3 \) is tame if it is locally tame except possibly at points of a closed subset \( X \) of \( S \) which is visible from a point of \( \text{Int } S \). Although Cobb’s work has not appeared, Burgess and Cannon [3, p. 327] have provided an outline of a proof. The techniques of their proof are also applicable to “seen” sets and will be used here. Daverman [6, Example 9.4] has an example of a wild \((n - 1)\)-sphere \( S \) in \( E^n \) \((n > 4)\) that is locally flat modulo a Cantor set visible from \( \text{Int } S \). The inflation technique described by Daverman in §9 of [6] has proven particularly useful for...
constructing in $E^n$ ($n > 4$) wild $(n - 1)$-spheres possessing certain geometric properties known to prohibit wildness if $n = 3$. In view of such examples (again, see [6, §9]) it may be somewhat surprising that the geometric property of visibility works for $n > 3$.

The paper concludes with a brief §4 in which two localized versions of the main theorems of §§2 and 3 are given. Though much more cumbersome to state, these local versions are established by the proofs given for their respective global analogues.

2. Flatness of the union of two cells. Theorem 1.2 of [7] states that a crumpled $n$-cube in $E^n$ is an $n$-cell if it is the union of two $n$-cells and if $n \neq 4$. Perhaps of importance, other then in its applicaton to visibility theorems in the next section, is the following variation of the result of [7]. The proof given for this theorem depends directly on Cannon's Invariance of 1-ULC [5, Theorem 4.2] and also establishes a more general local version given in §4. A crumpled $n$-cube is the closure of a bounded complementary domain of an $(n - 1)$-sphere topologically embedded in Euclidean $n$-space $E^n$.

Theorem 2.1. The interior of a crumpled $n$-cube $C$ in $E^n$ is 1-ULC if there exist two $n$-cells $C_1$ and $C_2$ in $C$ such that $\text{Bd } C \subset (\text{Bd } C_1) \cup (\text{Bd } C_2)$ and $C_2$ is flat in $E^n$. Furthermore $C$ is an $n$-cell if $n \neq 4$.

Proof. Let $S_1 = \text{Bd } C_1$, $S_2 = \text{Bd } C_2$, $S = \text{Bd } C$, $p \in S$, and let $\delta$ be a positive number. Since $C_1$ is an $n$-cell there is a positive number $\sigma$ such that each loop in $N(p, \sigma) \cap \text{Int } C_1$ can be shrunk to a point in $N(p, \delta) \cap \text{Int } C_1$. Since $S_2$ is flat it follows from Theorem 2C.6 of [5] (see [5, p. 61] for the case $n > 3$) that $E^n - S$ is 1-ULC in $E^n - S \cap S_2$. Consequently there is a positive number $\delta$ such that each loop in $N(p, \delta) \cap \text{Int } C$ can be shrunk to a point in $N(p, \delta) - S \cap S_2$. To show $\text{Int } C$ is 1-ULC it suffices to show that each loop in $N(p, \delta) \cap \text{Int } C$ can be shrunk to a point in $N(p, \delta) \cap \text{Int } C$.

Let $D$ denote a 2-cell, and let $f$ be a map taking $\text{Bd } D$ into $N(p, \delta) \cap \text{Int } C$. By the choice of $\delta, f$ extends to a map, which is still called $f$, of $D$ into $N(p, \sigma) - S \cap S_2$. The component $K$ of $D - f^{-1}(S \cap f(D))$ containing $\text{Bd } D$ contains a finite collection of disjoint simple closed curves whose union separates $\text{Bd } D$ from $f^{-1}(S \cap f(D))$. Since $f(D) \cap S_2 = \emptyset$ these curves can be chosen close enough to $f^{-1}(S \cap f(D))$ to insure that their images under $f$ lie in $\text{Int } C_1$. Moreover $f(K) \subset N(p, \sigma)$, so these simple closed curves in $K$ shrink to a point in $N(p, \delta) \cap \text{Int } C_1$. Consequently $f|\text{Bd } D$ can be extended to map all of $D$ into $N(p, \delta) \cap \text{Int } C$.

Bing [2] proved that $C$ is a 3-cell if its interior is 1-ULC. A discussion of the analogous result for $n > 5$ is contained in §5 of [6] where references are given.

Corollary 2.2. If $C_1$ and $C_2$ are $n$-cells in $S^n$ ($n \neq 4$) such that $C_2$ is flat and $S^n - (C_1 \cup C_2)$ has an $(n - 1)$-sphere $S$ as its boundary, then $S$ bounds an $n$-cell containing $C_1 \cup C_2$.

Theorem 2.3. A crumpled cube $C$ in $E^3$ is a tame 3-cell if there exist two tame 3-cells in $C$ such that $\text{Bd } C$ lies in the union of their boundaries.
Proof. From Theorem 2.1, $C$ is a 3-cell. Let $C_1$ and $C_2$ be two tame 3-cells as hypothesized. It follows from Theorem 4.1 of [4] that $C_1$ and $C_2$ are $\ast$-taming sets since they are tame. Then $C_1 \cup C_2$ is a $\ast$-taming set [4, Theorem 3.7(1)], and it follows that $C$ is tame.

Example 2.4. The failure of Theorem 2.3 for $n > 3$ can be demonstrated by letting $C^*$ be the inflation in $S^4$ of the Alexander Horned Cube as described by Daverman [6, §9], and then taking $C$ as the closure of $S^4 - C^*$.

Example 2.5. In Theorem 2.1 and its corollary it is essential that at least one of the two cells $C_1$ and $C_2$ be flat. To illustrate this one can take $C$ to be the Alford crumpled cube [1], although any crumpled cube whose boundary is locally tame modulo a subset of a simple closed curve $J$ in $\text{Bd} C$ will do, provided it is not a 3-cell. Let $D_1$ and $D_2$ be disks whose union is $\text{Bd} C$ and whose common boundary $J$ contains the wild points of $\text{Bd} C$. Of course each $D_i$ may be assumed locally polyhedral at its interior points, so each $\text{Int} D_i$ may be moved slightly toward $\text{Int} C$ to obtain a disk $D_i'$ such that $D_i \cap D_i' = J$. Then $D_i \cup D_i'$ bounds a 3-cell $C_i$ in $C$, and $\text{Bd} C \subset (\text{Bd} C_1) \cup (\text{Bd} C_2)$.

3. Applications to visibility. It is easy to see that an $(n - 1)$-sphere $S$ in $E^n$ is flat if $S$ is seen from some point of $\text{Int} S$. The next theorem generalizes this observation to the situation where $S$ is the union of two seen closed sets, however, it turns out that flatness from the seen side (except in $E^3$ where $S$ is tame from both sides) is all that can be expected. Example 3.3 illustrates this fact. Example 3.4 shows that there is no generalization to the case where $S$ is the union of three or more seen closed sets.

Lemma 3.1. If the closed subset $X$ of the $(n - 1)$-sphere $S$ in $E^n$ is seen from a point $x$ in $\text{Int} S$, then there is a flat $n$-cell $C$ in $S \cup \text{Int} S$ such that $x \in \text{Int} C$, $X \subset \text{Bd} C$, and $\text{Bd} C$ is visible from $x$.

Although the definitions of "seen" and "visible" differ slightly, the proof of Theorem 9.3.2 outlined in [3] also establishes Lemma 3.1.

Theorem 3.2. The interior of a crumpled $n$-cell $C$ in $E^n$ is 1-ULC if there exist closed sets $X_1$ and $X_2$ in $\text{Bd} C$ and points $x_1$ and $x_2$ in $\text{Int} C$ such that $\text{Bd} C = X_1 \cup X_2$ and each $X_i$ is seen from $x_i$ ($i = 1, 2$). Thus for $n \neq 4, C$ is an $n$-cell.

Proof. Let $S = \text{Bd} C$ and apply Lemma 3.1 twice to obtain two flat $n$-cells $C_1$ and $C_2$ in $C$ such that $S \subset (\text{Bd} C_1) \cup (\text{Bd} C_2)$. Then Theorem 3.2 follows immediately from Theorem 2.1.

Example 3.3. An example of a 4-cell $C$ in $E^4$ is now described so that $\text{Bd} C$ is the union of two closed sets seen from $\text{Int} C$ while $\text{Ext} \text{Bd} C$ is not 1-ULC. Thus one cannot conclude from the hypothesis of Theorem 3.2 that $\text{Bd} C$ is nice from both sides (except when $n = 3$ as demonstrated in the proof of Theorem 2.3).

The example $C$ is constructed by inflating the Alexander Horned cube $C^1$ in $E^3 \times 0$ to a 3-sphere $S$ in $E^4$ (see [6, p. 400]) and defining $C$ as $S \cup \text{Ext} S$. It is particularly easy to find the two seen closed sets if two points $x$ and $y$ are chosen above and below $C^1$, respectively, before inflating $C$, because then one may inflate.
toward these points rather than vertically. The two closed sets $X$ and $Y$ are just the upper and lower hemispheres, respectively, of $S$ since $X$ is seen from $x$ and $Y$ from $y$. Of course $C$ is not really a crumpled 4-cube but this can be corrected by carefully running a tube out close to $E^3 \times 0$ and then letting it balloon around $S$ so as to interchange the interior and exterior of $S$. This puts both $x$ and $y$ in Int $S$, but $X$ and $Y$ need to be enlarged to insure their union is still the entire sphere.

**Example 3.4.** There is a crumpled cube $C$ in $E^3$ such that its boundary is the union of three closed sets $X_1$, $X_2$ and $X_3$ where each $X_i$ is seen from a point $x_i$ in Int $C$, yet $C$ is not a 3-cell. In fact the Fox-Artin crumpled cube, whose complement is pictured in [3, p. 270], can be described to satisfy these conditions with the three points $x_i$ collinear.

The main idea for the description is given in §3 of [7]. Let $A'$ be an arc running through the double points of a regular projection $P$ of the Fox-Artin arc $F$ into the $xy$-plane $\pi$ (see Figure 4 of [3, p. 270]). There is a homeomorphism $h$ of $\pi$ onto itself taking $A'$ to an arc $A$ on the $x$-axis. Now an equivalent embedding of $F$ can be constructed by lifting small "over-arcs" of $h(P)$ into the vertical half-plane $\{(x,y,z) \mid z > 0\}$ and making small adjustments in the $xy$-plane. Thus we may assume $F$ lies on a 3-page book $B$ with two pages in $\pi$, binding $A$, and a third page in the $xy$-plane (see Figure 1 of [7]).

Let $C'$ be a cube such that $B$ divides $C'$ into three cubes $C'_1$, $C'_2$ and $C'_3$ each with vertical and horizontal sides. By drilling a small tapered hole in $C'$ along $F$ we obtain the Fox-Artin crumpled cube from $C'$. However before drilling choose three collinear points $x_1$, $x_2$ and $x_3$ in the interiors of $C'_1$, $C'_2$ and $C'_3$, respectively. Now the tube is drilled along $F$ creating grooves in the $C'_i$. Careful drilling consistent with the location of the $x_i$ will transform $C'$ to the desired crumpled cube $C$ and $C'_i$ to 3-cells $C_1$, $C_2$ and $C_3$ such that $\text{Bd} C_i$ is seen from $x_i$. The closed sets $X_i$ are taken as $(\text{Bd} C_i) \cap (\text{Bd} C)$.

### 4. Local versions of the main theorems

The proofs given for Theorems 2.1 and 3.2 actually prove the more general theorems given in this section. These localized versions will be needed in subsequent work.

**Theorem 4.1.** The interior of a crumpled $n$-cell $C$ in $E^n$ is 1-LC at each point in the interior of $(n - 1)$-cell $D$ in $\text{Bd} C$ if there exist two $n$-cells $C_1$ and $C_2$ in $C$ such that $D \subset (\text{Bd} C_1) \cup (\text{Bd} C_2)$ and $C_1$ is flat in $E^n$. Furthermore $\text{Bd} C$ is locally flat at each point of $\text{Int} D$ if $n \neq 4$.

**Theorem 4.2.** The interior of a crumpled $n$-cell $C$ is 1-LC at each point in the interior of an $(n - 1)$-cell $D$ in $\text{Bd} C$ if there exist two closed sets $X_1$ and $X_2$ and two points $x_1$ and $x_2$ in $\text{Int} C$ such that $D = X_1 \cup X_2$ and $X_i$ is seen from $x_i$ ($i = 1, 2$). For $n \neq 4$ it follows that $\text{Bd} C$ is locally flat at each point of $\text{Int} D$.

**References**


DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84321 (Current address)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916