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A NOTE ON THE BOREL FORMULA

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ABSTRACT. A new proof of the Borel formula is obtained for $G = (Z_p)^r$ actions on spaces X having $H_i(X; Z_p) = 0$, $i \neq n$ (some n) and $H_n(X; Z_p) = Z_p \oplus$ Free Z_pG module. Each X^H must be a Z_p-homology $n(H)$-sphere and then $n - n(G) = \Sigma(n(H) - (G))$, sum running over corank 1 subgroups. A discussion of examples follows.

1. Introduction. If an elementary abelian p-group $G = (Z_p)^r$ acts on a Z_p-homology n-sphere, then Smith theory [3, Chapter 3] says that the fixed point set of a subgroup H of G is a Z_p-homology $n(H)$-sphere. If p is odd $n - n(H)$ must be even for all H. The Borel formula [2, p. 175] states that, moreover, $n - n(G) = \Sigma(n(H) - n(G))$ where the sum runs over all subgroups H of G such that $G/H = Z_p$ (i.e. H has corank 1).

In [4] the following converse result was established.

THEOREM A. *Let X be a finite CW complex with a cellular action of $G = (Z_p)^r$, such that each X^H $(H \neq 0)$ is a Z_p-homology $n(H)$-sphere. Suppose, moreover, that there exists an n such that $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$, and so that if p is odd, $n - n(G)$ is even. Also assume the Borel formula holds for this action (and n). Then $H_n(X; Z_p) = Z_p \oplus F$ where F is Z_pG free.*

This result was established by constructing a linear action of G on S^n and an equivariant map $\phi: X \to S^n$ which induces Z_p-homology isomorphisms $\phi^H: X^H \to (S^n)^H$, $H \neq 0$, and a Z_p-homology epimorphism when $H = 0$.

In the present paper we establish the converse to Theorem A, which provides an extension of the Borel formula. I am indebted to Professor Glen Bredon, who first suggested the approach I used here, and to Professor Gary C. Hamrick for many helpful conversations.

2. We begin with

PROPOSITION 1. *Let $G = (Z_p)^r$ $(r > 1)$ act cellularly on X, a finite CW complex such that each X^H $(H \neq 0)$ is a Z_p-homology $n(H)$-sphere. Suppose there is an integer n so that $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$, and $H_n(X; Z_p) = Z_p \oplus F$, where F is free over Z_pG. Then the Borel formula holds, i.e.,*

$$n - n(G) = \sum (n(H) - n(G)),$$

sum extending over all corank 1 subgroups H in G.

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PROOF. Let k be the cellular dimension of X. The proof proceeds in roughly two stages. At first we construct inductively a finite CW complex, Y_{k-1}, with the property that Y_{k-1} is k-dimensional, $k-1$-connected. Y_{k-1} will also have the property that

$$\operatorname{Ext}^{s+(k-n)}_{Z_pG}\big(H_n(X; Z_p), Z_p\big) = \operatorname{Ext}^{s}_{Z_pG}\big(H_k(Y_{k-1}; Z_p), Z_p\big)$$

for all $s \geqslant 2$.

On the other hand we define a finite CW complex $\hat{Y}_{k-1}$ which satisfies the hypothesis of Theorem A in the introduction, for the integer $m = \Sigma(n(H) - n(G)) + n(G)$ (not, at this point, necessarily equal to n). It will result that

$$\operatorname{Ext}^{s+(k-n)}_{Z_pG}\big(H_n(X; Z_p), Z_p\big) = \operatorname{Ext}^{s+(k-m)}_{Z_pG}\big(H_m(\hat{Y}_{k-1}; Z_p), Z_p\big)$$

for any $s \geqslant 2$. Since G has nonperiodic group cohomology for $r > 1$, it follows that $n = m$, which is what we wanted to show.

To begin with, we will use extensively the fact that whenever $f\colon M \to T \oplus P$ is a ZG epimorphism of ZG modules with M free, P projective and T consisting of torsion prime to p, $\ker f$ is ZG projective. This follows from [7, 4.12] once it is clear that T is cohomologically trivial. But multiplication by $|G|$ on T is an automorphism, which induces an automorphism on cohomology. Coupling this with the fact that $H^*(G, \cdot)$ has exponent $|G|$ (see Cartan-Eilenberg, Chapter XII, e.g.), it follows that T is cohomologically trivial (this argument was pointed out to me by R. Oliver).

We may assume X is 1-connected (by suspending). Since $\tilde{H}_i(X; Z)$ is torsion prime to p, we may attach free orbits of cells in dimensions $\leqslant n$ to produce the complex Y such that Y is $n-1$-connected and $H_n(Y; Z) = H_n(X; Z) \oplus N$, N a ZG-projective (for a nice exposition of the procedure of "equivariantly attaching cells", see [5, §1]). Now add free orbits of $n+1$-cells to kill off N. This produces a complex X_0 with $H_n(X_0; Z) = H_n(X; Z)$ and $H_{n+1}(X_0; Z) = H_{n+1}(X; Z) \oplus N_0$, N_0 a ZG-projective. $H_i(X_0; Z) = H_i(X; Z)$, $i > n+1$.

Add free orbits of $n+1$-cells to X_0 to kill off $H_n(X; Z)$. This creates an n-connected, k-dimensional complex Y_n, so that $H_i(Y_n) = H_i(X)$, $i > n+1$, and

$$0 \to H_{n+1}(X_0; Z) \to H_{n+1}(Y_n; Z) \to \operatorname{Ker} \partial_{n+1} \to 0 \tag{$*$}$$

where $0 \to \operatorname{Ker} \partial_{n+1} \to H_{n+1}(Y_n, X_0; Z) \to H_n(X_0; Z) \to 0$. Now if $k = n+1$, it follows that $H_{n+1}(X_0; Z) = N_0$, since $H_{n+1}(X; Z_p) = 0$. By [7, 3.5], $N_0 \otimes Z_p$ is Z_pG free. After tensoring the above sequences with Z_p, it follows that

$$\operatorname{Ext}^{s}_{Z_pG}\big(H_n(X; Z_p), Z_p\big) = \operatorname{Ext}^{s-1}_{Z_pG}\big(H_{n+1}(Y_n; Z_p), Z_p\big) \quad \text{for any } s \geqslant 2.$$

If $k > n+1$, we show below how to construct Y_{n+1}, which is $n+1$-connected and k-dimensional, from Y_n. If need be, the following argument may be repeated until one obtains Y_{k-1}.

Add free orbits of $n+2$-cells to Y_n to kill off $H_{n+1}(X_0; Z)$ (see sequence $(*)$). This produces a complex Y'_{n+1} with $H_{n+1}(Y'_{n+1}; Z) = \operatorname{Ker} \partial_{n+1}$ and $H_{n+2}(Y'_{n+1}; Z) = H_{n+2}(X_0; Z) \oplus N_1$, where N_1 is ZG-projective.

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Now add free orbits of $n+2$-cells to Y'_{n+1} to form Y_{n+1} and to kill off Ker ∂_{n+1}. Y_{n+1} is $n+1$-connected, k-dimensional. Further, we have:

$$H_{n+2}(Y_{n+1}, Y'_{n+1}; Z) \to H_{n+1}(Y'_{n+1}, X_0; Z) \xrightarrow{\partial_{n+1}} H_n(X_0; Z) \to 0$$

where the left-hand map is the composition

$$H_{n+2}(Y_{n+1}, Y'_{n+1}; Z) \twoheadrightarrow H_{n+1}(Y'_{n+1}; Z) \approx \operatorname{Ker} \partial_{n+1} \rightarrowtail H_{n+1}(Y'_{n+1}, X_0; Z).$$

We also have the exact sequence,

$$\begin{aligned} 0 \to H_{n+2}(Y'_{n+1}; Z) &\to H_{n+2}(Y_{n+1}; Z) \\ &\to H_{n+2}(Y_{n+1}, Y'_{n+1}; Z) \overset{\partial_{n+2}}{\twoheadrightarrow} H_{n+1}(Y'_{n+1}; Z). \end{aligned}$$

Now if $k = n+2$, it follows that $H_{n+2}(Y'_{n+1}; Z) = N_1$, a ZG-projective, since $H_{n+2}(X; Z_p) = 0$. Thus $H_{n+2}(Y'_{n+1}; Z_p)$ is Z_pG free. As before we have the exact sequences

$$0 \to H_{n+2}(Y'_{n+1}; Z_p) \to H_{n+2}(Y_{n+1}; Z_p) \twoheadrightarrow \operatorname{Ker} \partial_{n+2} \otimes Z_p,$$

$$0 \to \operatorname{Ker} \partial_{n+2} \otimes Z_p \to H_{n+2}(Y_{n+1}, Y'_{n+1}; Z_p) \twoheadrightarrow H_{n+1}(Y'_{n+1}; Z_p).$$

Then

$$\operatorname{Ext}^s_{Z_pG}\big(H_{n+1}(Y'_{n+1}; Z_p), Z_p\big) = \operatorname{Ext}^{s-1}_{Z_pG}\big(H_{n+2}(Y_{n+1}; Z_p), Z_p\big)$$

and

$$\operatorname{Ext}^s_{Z_pG}\big(H_n(X; Z_p), Z_p\big) = \operatorname{Ext}^{s-1}_{Z_pG}\big(H_{n+1}(Y'_{n+1}; Z_p), Z_p\big)$$

since $H_{n+1}(Y'_{n+1}; Z_p) = \operatorname{Ker} \partial_{n+1} \otimes Z_p$ and where s is arbitrary, $s \geqslant 2$.

If $k > n+2$, we have the exact sequence

$$0 \to H_{n+2}(Y'_{n+1}; Z) \to H_{n+2}(Y_{n+1}; Z) \twoheadrightarrow \operatorname{Ker} \partial_{n+2}.$$

Since $H_{n+2}(Y'_{n+1}; Z) = H_{n+2}(X_0; Z) \oplus N_1$, we can repeat the procedure above to reach, eventually, Y_{k-1}. By adding additional free orbits of cells, if necessary, we may assume $k - m \geqslant 2$.

The same argument as above shows that for any $s \geqslant 2$,

$$\operatorname{Ext}^{s+(k-n)}_{Z_pG}\big(H_n(X; Z_p), Z_p\big) = \operatorname{Ext}^s_{Z_pG}\big(H_k(Y_{k-1}; Z_p), Z_p\big).$$

Now, on the other hand, consider $\hat{Y}_{k-1} = Y^{(m)}_{k-1} \cup Y^F_{k-1}$, where $Y^F_{k-1} = \bigcup_{H \neq 0} Y^H_{k-1} = \bigcup_{H \neq 0} X^H$. We claim that $\tilde{H}_i(\hat{Y}_{k-1}; Z_p) = 0$ for $i \neq m$. This amounts to showing that $H_*(\bigcup_{H \neq 0} X^H; Z_p) = 0$ for $* > m$. First of all, by induction on the rank of G, r, we may assume the Borel formula for any elementary abelian p-group of lesser rank. This induction implies that $n(H) \leqslant m$, for $H \neq 0$. Letting S be any nonempty collection of nonzero subgroups of G (without loss, not containing G itself) one actually establishes the stronger fact that $H_*(\bigcup_{H \in S} X^H; Z_p) = 0$ if $* > m$. To show this one uses induction on the cardinality of S and Mayer-Vietoris. Thus one is led to see it suffices to show that, for $M_0 \in S$,

$$H_*\Big(X^{M_0}, \bigcup X^{MM_0}; Z_p\Big) = 0, \quad \text{for } * > n(M_0),$$

where the union is over $S - \{M_0\}$. This latter fact is established by a double induction on rank M_0 and on the cardinality of $S - \{M_0\}$.

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There are two naturally arising cases. If there is some M_1 in $S - \{M_0\}$ with $n(M_1M_0) < n(M_0)$ one considers the homology sequence of the triple $(X^{M_0}, \bigcup X^{MM_0}, \bigcup X^{MM_0})$, where the first union is over $S - \{M_0\}$ and the second is over $S - \{M_0, M_1\}$. An application of excision establishes the result in this case.

The second case is that every $M \in S$ has $n(MM_0) = n(M_0)$. Here, it suffices to show that if M_1 and M_2 both have $n(M_1M_0) = n(M_2M_0) = n(M_0)$ then $n(M_0) = n(M_1M_2M_0)$ (by a Mayer-Vietoris argument). To demonstrate this last property use the fact that the Borel formula holds whenever a group of rank $\leqslant r - 1$ is acting.

Now since $\tilde{H}_i(\hat{Y}_{k-1}; Z_p) = 0$, $i \neq m$, and the Borel formula holds on $\hat{Y}_{k-1}$, by Theorem A in the introduction (it is not hard to see that if $r > 1$, $m - n(G)$ is always even if p is odd)

$$H_m(\hat{Y}_{k-1}; Z_p) = Z_p \oplus F_1,$$

where F_1 is a free Z_pG module.

Now consider $C_*(Y_{k-1}, \hat{Y}_{k-1}; Z_p)$. We have the exact sequence,

$$0 \to H_k(Y_{k-1}; Z_p) \to C_k(Y_{k-1}, \hat{Y}_{k-1}; Z_p) \to \dots$$
$$\to C_{m+1}(Y_{k-1}, \hat{Y}_{k-1}; Z_p) \twoheadrightarrow H_m(\hat{Y}_{k-1}; Z_p).$$

Since each $C_i(Y_{k-1}, \hat{Y}_{k-1}; Z_p)$ is Z_pG free, it follows that for $s \geqslant 2$,

$$\operatorname{Ext}^{s+(k-m)}_{Z_pG}(H_m(\hat{Y}_{k-1}; Z_p), Z_p) = \operatorname{Ext}^s_{Z_pG}(H_k(Y_{k-1}; Z_p), Z_p).$$

Thus,

$$\operatorname{Ext}^{s+(k-n)}_{Z_pG}(H_n(X; Z_p), Z_p) = \operatorname{Ext}^{s+(k-m)}_{Z_pG}(H_m(\hat{Y}_{k-1}; Z_p), Z_p).$$

But since $H_n(X; Z_p)$ and $H_m(\hat{Y}_{k-1}; Z_p)$ are both stably Z_p, the result is that

$$\operatorname{Ext}^{s+(k-n)}_{Z_pG}(Z_p, Z_p) = \operatorname{Ext}^{s+(k-m)}_{Z_pG}(Z_p, Z_p).$$

This implies that $n = m$, since G has nonperiodic cohomology. □

3. We now consider examples of Proposition 1 which are n-dimensional, $n - 1$ connected.

PROPOSITION 2. *Suppose that X satisfies the hypotheses of Proposition* 1 *and that X is n-dimensional, $n - 1$ connected. Then $H_n(X; Z) = Z \oplus P$, where P is ZG-projective and Z has trivial G-action if p is odd.*

PROOF. It was shown in [4] that there is a linear G-action on S^n and an equivariant map ϕ: $X \to S^n$ having the properties described in the introduction. From the mapping cone C_ϕ, we have the exact sequence

$$0 \to H_{n+1}(C_\phi; Z) \to H_n(X; Z) \xrightarrow{\phi_*} H_n(S^n; Z) \to H_n(C_\phi; Z) \to 0.$$

Because ϕ_* is a Z_p-homology epimorphism we have

$$0 \to H_{n+1}(C_\phi; Z) \to H_n(X; Z) \twoheadrightarrow \operatorname{Im} \phi_* \approx Z \quad \text{(with some action).} \qquad (*)$$

Also, $H_n(C_\phi; Z) = Z_k$, where $(k, p) = 1$ and $\operatorname{Im} \phi_* = (k)$. One may add free orbits of $n + 1$ cells to C_ϕ to kill $H_n(C_\phi; Z)$. This results in an $(n + 1)$-dimensional,

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n-connected G-complex Y such that $H_{n+1}(Y; Z) = H_{n+1}(C_\phi; Z) \oplus P$, P a projective ZG module. By [**6**, Lemma 3], $H_{n+1}(Y; Z)$ is ZG-projective and then so is $H_{n+1}(C_\phi; Z)$.

If Im $\phi_* \approx Z$ has trivial G-action then one has

$$\operatorname{Ext}_{ZG}(Z; H_{n+1}(C_\phi; Z)) = H^1(G; H_{n+1}(C_\phi; Z)) = 0$$

by [**6**, 4.11]. Thus in this case, $(*)$ splits.

If Im $\phi_* \approx Z$ has nontrivial G-action (denoted Z^-), it follows that $Z^- \otimes_Z Z^- \approx Z$ has trivial (diagonal) G-action and that whenever P is ZG-projective, so is $Z^- \otimes_Z P$ (diagonal action). This latter statement follows from the fact that $Z^- \otimes_Z ZG$ (diagonal action) is isomorphic, as ZG modules, to ZG. Now tensoring $(*)$ over Z with Z^- one has the exact sequence of ZG modules,

$$0 \to H_{n+1}(C_\phi; Z) \otimes_Z Z^- \to H_n(X; Z) \otimes_Z Z^- \to Z \to 0. \qquad (*)'$$

By the argument above, $(*)'$ splits. So,

$$H_n(X; Z) \otimes_Z Z^- \approx Z \oplus H_{n+1}(C_\phi; Z) \otimes_Z Z^-.$$

Now, tensoring again with Z^- over Z, one has $H_n(X; Z) \approx Z^- \oplus H_{n+1}(C_\phi; Z)$. $\square$

As an example, one may take the wedge of a standard linear action on S^n with a bouquet of n-spheres permuted freely by G. On the other hand, suppose Y is an n-dimensional $n-1$-connected CW complex with G-action such that Y^H is Z_p-acyclic for each $H \lhd G$, $H \neq 0$. By [**6**, Lemma 3], $H_n(Y)$ is ZG-projective. Wedging Y with a linear S^n would yield other examples. However, it is a result of Swan (see [**1**, p. 178]) that $H_n(Y)$ must be stably free, when $G = Z_p$. Consequently, one can construct projective modules P over ZG, $G = (Z_p)^r$, $r > 1$, which could never be realised as an $H_n(Y)$, as above. Mainly, this is so because a necessary condition on P is that the image of P in $G_0(ZG)/S$ be zero, where $G_0(ZG)$ is the Grothendieck group on finitely generated ZG modules and S is the subgroup generated by signed permutation modules (see [**1**]). As a result it is not clear precisely when a projective ZG module can be an $H_n(Y)$, as above. According to [**1**] the necessary condition above is sufficient for $G = Z_p$.

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