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TOPOLOGICAL ENTROPY OF BLOCK MAPS

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ABSTRACT. We show that $h(f_\infty) = \log 2$ where f_∞ is the map on the space of sequences of zeros and ones induced by the block map $f(x_0, \ldots, x_k) = x_0 + \prod_{i=1}^k (x_i + b_i)$ where $k \geq 2$ and the k-block $b_1 \ldots b_k$ is aperiodic.

1. Introduction. Topological entropy, a conjugacy invariant of continuous self-maps of compact Hausdorff spaces, was introduced in **[AKM]** in 1965. Over the years it has become an important concept in both topological and differentiable dynamics. For any excellent account of its place in present-day dynamics, see **[B3]**.

Exact computations of topological entropy, other than for maps with zero entropy, appear to be rare. Exceptions are the Chebyshev polynomials **[AM]**, endomorphisms of Lie groups **[B2]**, Axiom A diffeomorphisms **[B1]**, and a number of classes of subshifts. More common are results giving bounds for entropy, e.g., the results dealing with Shub's "entropy conjecture" [**B3**, Chapter 5], the recent results for maps of the interval **[BF]**, **[JR]**.

In this paper we will compute the topological entropy of a class of shift-commuting maps of the space X of one-sided sequences of zeros and ones. In particular, we will prove the following.

THEOREM. *Let* $f(x_0, \ldots, x_k) = x_0 + \prod_{i=1}^k (x_i + b_i)$, *where* $k \geq 2$ *and the* k*-block* $B = b_1 \ldots b_k$ *is aperiodic. Then* $h(f_\infty) = \log 2$.

Here the arithmetic is to be done in GF(2), $f_\infty: X \to X$ is defined by $[f_\infty(x)]_i = f(x_i, \ldots, x_{i+k})$ and B is aperiodic means that there is no p, $1 \leq p \leq k-1$, such that $b_i = b_{i+p}$ for $1 \leq i \leq k-p$.

It is rather surprising that, despite the finite nature of these maps [**H**, Theorem 3.4], there have been no previous entropy computations for shift-commuting maps.

2. Preliminaries. We assume the reader is familiar with the elementary properties of topological entropy, denoted $h(\,)$.

Let X denote the set of all sequences $x = x_0x_1x_2 \ldots$ where each $x_i = 0$ or 1. Thus $X = \prod_0^\infty \{0, 1\}$. We give $\{0, 1\}$ the discrete topology and X the product topology. Then X is a compact, metrizable space, homeomorphic to the Cantor set. A neighborhood base at $x \in X$ consists of all sets of the form $\{y \in X \mid y_0 \ldots y_n = x_0 \ldots x_n\}$ where $n \geq 0$.

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An *n-block* is a concatenation of n zeros and ones, i.e., a member of $\{0, 1\}^n$. An *n-block map* is a function f: $\{0, 1\}^n \to \{0, 1\}$. Such a function also maps $(n + 1)$-blocks to 2-blocks by

$$f(x_0, \ldots, x_n) = (f(x_0, \ldots, x_{n-1}), f(x_1, \ldots, x_n)).$$

Similarly, f maps $(n + 2)$-blocks to 3-blocks, etc., and f induces a continuous map f_∞: $X \to X$ defined by $[f_\infty(x)]_i = f(x_i, \ldots, x_{i+n-1})$. The *shift* σ: $X \to X$, defined by $[\sigma(x)]_i = x_{i+1}$, is induced by the 2-block map $s(x_0, x_1) = x_1$. It is well known that $h(\sigma) = \log 2$. The set of continuous, shift-commuting maps of X to itself coincides with $\{f_\infty | f \text{ is an } n\text{-block map}, n \geqslant 1\}$ [**H**, Theorem 3.4].

For the purposes of evaluating block maps, it is convenient to think of the symbol set $\{0, 1\}$ as GF(2), the field with two elements. The set of n-block maps coincides with the set of polynomials in n variables over GF(2) of degree at most one in each variable [**H**, Theorem 19.1].

Composition of block maps is defined so that $(g \circ f)_\infty = g_\infty \circ f_\infty$. For example, if f is a $(k + 1)$-block map, then f^2 is the $(2k + 1)$-block map defined by

$$f^2(x_0, \ldots, x_{2k}) = f(f(x_0, \ldots, x_k), \ldots, f(x_k, \ldots, x_{2k})).$$

Let f be a $(k + 1)$-block map such that $f(x_0, \ldots, x_k) = g(x_0, \ldots, x_{k-1}) + x_k$ for some k-block map g. Then f_∞ is conjugate to σ^k, the conjugacy being given by $[\psi(x)]_{nk+j} = f^n(x_j, \ldots, x_{nk+j})$. Hence in this case $h(f_\infty) = k\log 2$. In fact, it can be shown that for any $(k + 1)$-block map f, $h(f_\infty) \leqslant k\log 2$ with equality if and only if $f(x_0, \ldots, x_k) = g(x_0, \ldots, x_{k-1}) + x_k$ for some k-block map g.

3. The result. Let f be a $(k + 1)$-block map of the form

$$f(x_0, \ldots, x_k) = x_0 + \prod_{i=1}^{k} (x_i + b_i + 1)$$

where $k \geqslant 2$ and the k-block $B = b_1 \ldots b_k$ is aperiodic, i.e., there is no p, $1 \leqslant p \leqslant k - 1$, such that $b_i = b_{i+p}$ for $1 \leqslant i \leqslant k - p$. We will prove that $h(f_\infty) = \log 2$.

These block maps were studied by the author and G. A. Hedlund in [**CH**] where they were used as feedback functions for nonlinear shift registers. The maps f_∞ under consideration are continuous, finite-to-one [**H**, Theorem 5.5], map X onto X [**H**, Theorem 6.6] and commute with the shift. However they are not transitive, i.e., there is no point $x \in X$ with a dense f_∞-orbit.

In the sequel, we will use juxtaposition of blocks to denote concatenation, omitting parentheses and commas. For example, we will write an expression such as $f(CD) = E$ when C, D and E are k-blocks.

The following notation from [**CH**] will prove helpful. Let $\tilde{0} = 1$, $\tilde{1} = 0$ and for an n-block $A = a_1 \ldots a_n$ with $n \geqslant 2$, let $\tilde{A} = a_1 \ldots a_{n-1}\tilde{a}_n$. Then by [**CH**, Lemma 6]

(i) B is not an "interior block" for any of BB, $B\tilde{B}$, $\tilde{B}B$ or $\tilde{B}\tilde{B}$.

We collect together below some useful facts about the block map f. See [**CH**, Lemmas 7 and 9].

(ii) If $f(CD) = B$, then $C = B$ or $\tilde{B}$.

(iii) $f(CD) = \tilde{C}$ if and only if $D = B$.

(iv) $f(CB) = \tilde{B}$ if and only if $C = B$.

(v) $f(CB) = B$ if and only if $C = \tilde{B}$.

(vi) If B is not an interior block of CD, then $f(CD) = C$ or $\tilde{C}$.

Let $X_B = \{x \in X \mid \text{each } x_{ik} \ldots x_{ik+k-1} = B \text{ or } \tilde{B}\}$, the set of concatenations of B's and $\tilde{B}$'s. Then X_B is closed and it follows from (i) and (vi) that f_∞ maps X_B to itself. Furthermore, $f_\infty | X_B$ is conjugate to g_∞ where $g(x_0, x_1) = x_0 + x_1$, the conjugacy being given by

$$[\varphi(x)]_i = \begin{cases} 1 & \text{if } x_{ik} \ldots x_{ik+k-1} = B, \\ 0 & \text{if } x_{ik} \ldots x_{ik+k-1} = \tilde{B}. \end{cases}$$

Since g_∞ is conjugate to the shift σ, it follows that $h(f_\infty | X_B) = \log 2$ and hence that $h(f_\infty) \geqslant \log 2$. We will show that $h(f_\infty) \leqslant \log 2$ by showing that for each $x \in X$, $h(f_\infty | \text{cl } \mathcal{O}(x)) \leqslant \log 2$, where $\mathcal{O}(x)$ denotes the f_∞-orbit of x, $\{f_\infty^n(x) | n = 0, 1, \ldots\}$. The result then follows from [**G**, Corollary 1].

Case 1. $x \in X_B$. Then $h(f_\infty | \text{cl } \mathcal{O}(x)) \leqslant h(f_\infty | X_B) = \log 2$.

Case 2. B appears infinitely often in x but $\sigma^n(x) \notin X_B$ for all $n \geqslant 0$. Write $x = A_1 C_1 A_2 C_2 \ldots$ using the following procedure. We illustrate the procedure for $B = 011$ and $x = 10011010010110110101010011010011 00 \ldots$.

Step 1. Underline the occurrences of B in x.

$$x = 10\underline{011}\ 01001\underline{011}\ \underline{011}\ 0101010\underline{011}\ 010\underline{011}\ 00\ldots.$$

Step 2. For each occurrence of B in x, underline the maximal concatenation of B's and $\tilde{B}$'s which ends in the indicated occurrence of B.

$$x = 10\underline{011}\ 01001\underline{\underline{011}\ 011}\ 0101\underline{\underline{010011}\ 010011}\ 00\ldots.$$

Step 3. For each concatenation in Step 2 which is not a subconcatenation of another concatenation in Step 2, underline the maximal concatenation of B's and $\tilde{B}$'s which can be obtained by extending to the right without overlapping the next concatenation.

$$x = 10\underline{011010}\ 01\underline{011011010}\ 1\underline{010011010011}\ 00\ldots.$$

Step 4. Label the underlined concatenations of Step 3 by $C_1, C_2, \ldots$ and label the nonunderlined block preceding C_i by A_i.

$$x = \underset{A_1}{\underleftrightarrow{10}}\ \underset{C_1}{\underleftrightarrow{011010}}\ \underset{A_2}{\underleftrightarrow{01}}\ \underset{C_2}{\underleftrightarrow{011011010}}\ \underset{A_3}{\underleftrightarrow{1}}\ \underset{C_3}{\underleftrightarrow{010011010011}}\ 00\ldots.$$

Note that in our example, $C_1 = 011010 = B\tilde{B}$ is followed by $010 = \tilde{B}$, but that this $\tilde{B}$ did not get underlined in Step 3, for otherwise C_1 and C_2 would overlap.

The decomposition $x = A_1 C_1 A_2 C_2 \ldots$ has the following properties.

(1) $A_i \neq \varnothing$ if $i \geqslant 2$.

(2) B does not appear in A_i.

(3) A_i does not begin with $\tilde{B}$ if $i \geqslant 2$.

(4) A_i does not end with $\tilde{B}$.

(5) C_i is a concatenation of B's and $\tilde{B}$'s.

Now write $f_\infty(x) = A_1^1 C_1^1 A_2^1 C_2^1 \ldots$ where A_i^1 has the same length as A_i and C_i^1 has the same length as C_i. In this case we say that "A_i appears above A_i^1", etc. The meaning of the phrase "D appears above E" in similar situations will be clear from context.

PROPOSITION. *The decomposition* $f_\infty(x) = A_1^1C_1^1A_2^1C_2^1 \ldots$ *also has properties* (1)–(5).

PROOF. Property (1) is clear.

(2) Suppose B appears in A_i^1. Then by (iii), either B or $\tilde{B}$ appears above B. Since B does not appear in A_i, $\tilde{B}$ must appear above B. Then by (iii), the k-block in x immediately following this appearance of $\tilde{B}$ is B. Thus $\tilde{B}B$ appears in A_i or in A_iC_i. Since B does not appear in A_i, this appearance of B must be entirely in C_i. Then A_i ends with $\tilde{B}$, contrary to (4).

(3) Let $i \geqslant 2$ and suppose A_i^1 begins with $\tilde{B}$. Let D be the k-block in x above this appearance of $\tilde{B}$ and let E be the k-block in x immediately following this appearance of D. Then A_i begins with D and so, by (2), $D \neq B$ and, by (3), $D \neq \tilde{B}$. Then by (iv), B is an interior block of DE. This appearance of B must be entirely in C_i and hence A_i and C_i overlap. Thus A_i^1 does not begin with $\tilde{B}$.

(4) Suppose A_i^1 ends with $\tilde{B}$. Since C_i begins with B or $\tilde{B}$, by (iv) and (v), A_i ends with B or $\tilde{B}$, contrary to (2) or (4).

(5) Let $C_i = B_1 \ldots B_m$ where each $B_j = B$ or $\tilde{B}$ and let D be the initial k-block of $A_{i+1}C_{i+1}$. Then by (i) and (vi), $C_i^1 = B_1^1 \ldots B_{m-1}^1E$ where each $B_j^1 = B$ or $\tilde{B}$ and $E = f(B_mD)$. But B is not an interior block of B_mD, for otherwise C_i and C_{i+1} would overlap. Then by (vi), $E = B_m$ or $\tilde{B}_m$, i.e., $E = B$ or $\tilde{B}$. □

Since the proof of the proposition involved only the properties of the original decomposition and not the procedure used to obtain them, it follows that for each $n \geqslant 1$, the decomposition $f_\infty^n(x) = A_1^nC_1^nA_2^nC_2^n \ldots$, where A_i^n has the same length as A_i and C_i^n has the same length as C_i, also has properties (1)–(5).

Define $A_i^1 = A_i$ and $C_i^0 = C_i$. Let i be fixed and consider the sequences of blocks $\{A_i^0, A_i^1, \ldots\}$ and $\{C_i^0, C_i^1, \ldots\}$.

Let D^n be the terminal k-block of C_i^n and let E^n be the initial k-block of $A_{i+1}^nC_{i+1}^n$. Then B is not an interior block of D^nE^n, so by (vi), $D^{n+1} = D^n$ or $\widetilde{D^n}$. By (2), $E^n \neq B$, so by (iii), $D^{n+1} = D^n$. Thus the terminal k-block of C_i^n is the same for all n and therefore the sequence $\{C_i^0, C_i^1, \ldots\}$ is periodic, say with period q_i.

Let F^n be the initial k-block of C_i^n. Since B does not appear in any A_i^n and B is not an interior block of $A_i^nF^n$, it follows from (vi) and (iii) that $A_i^{n+1} = A_i^n$ or $\widetilde{A_i^n}$. Therefore the sequence $\{A_i^0, A_i^1, \ldots\}$ is periodic, with (not necessarily least) period $p_i = 2q_i$.

Define $D_i^n = A_i^nC_i^n$. Then the sequence of blocks $\{D_i^0, D_i^1, \ldots\}$ is periodic with period p_i. Since $f_\infty^n(x) = D_1^nD_2^n \ldots$ and D_i^n appears above D_i^{n+1}, it follows that $f_\infty|\text{cl } \mathcal{O}(x)$ is conjugate to a rotation on the compact group $\mathbf{Z}_{p_1} \times \mathbf{Z}_{p_2} \times \cdots$ and hence $h(f_\infty|\text{cl } \mathcal{O}(x)) = 0$.

Case 3. $x \notin X_B$ but $\sigma^n(x) \in X_B$ for some $n \geqslant 1$. Then, in a manner similar to Case 2, $f_\infty|\text{cl } \mathcal{O}(x)$ is conjugate to the product of a rotation on a finite group and $f_\infty|\text{cl } \mathcal{O}(y)$ for some $y \in X_B$. Hence $h(f_\infty|\text{cl } \mathcal{O}(x)) = h(f_\infty|\text{cl } \mathcal{O}(y)) \leqslant \log 2$.

Case 4. B appears only finitely often in x. Then $f_\infty|\mathrm{cl}\ \mathcal{O}(x)$ is conjugate to a rotation on a finite group and hence $h(f_\infty|\mathrm{cl}\ \mathcal{O}(x)) = 0$.

Finally, by [**G**, Corollary 1], $h(f_\infty) = \sup_{x \in X} h(f_\infty|\mathrm{cl}\ \mathcal{O}(x)) \leqslant \log 2$, so $h(f_\infty) = \log 2$.

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