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RETRACTS IN METRIC SPACES

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ABSTRACT. In this paper we define S-contractibility and two classes of spaces connected with this notion. A space X is said to be S-contractible provided that S is a function $S: X \times \langle 0, 1\rangle \times X \ni (x, \alpha, y) \mapsto S_x(\alpha, y) \in X$ that is continuous in α and y, and for every $x, y \in X$, $S_x(0, y) = y$, $S_x(1, y) = x$. This notion is close to equiconnectedness, which can be defined as follows. A space X is equiconnected if there exists a map S such that X is S-contractible and $S_x(\alpha, x) = x$ for all $x \in X$ and $\alpha \in I$ (cf. [4]). The results we obtain in the theory of retracts are close to those that are known for equiconnected spaces. Also the thickness of the neighborhood that can be retracted on a set in a metric space is estimated, which enables to prove a theorem belonging to fixed point theory.

1. We repeat the notions related to equiconnectedness [2].

DEFINITIONS. A *local equiconnecting function* for a space X is a map $\lambda: U \times I \to X$, where U is a neighborhood of the diagonal in $X \times X$ such that $\lambda(x_0, x_1, i) = x_i$, $i = 0, 1$, and $\lambda(x, x, t) = x$ for every $x_0, x_1, x \in X$, $t \in I$.

The λ-*extension* of a subset $A \subset X$ is the smallest nonempty subset $\hat{A} \subset X$ (if it exists) such that $A \times \hat{A} \subset U$ and $\lambda(A \times \hat{A} \times I) \subset \hat{A}$. A is λ-*convex* if $A = \hat{A}$.

A local equiconnecting function λ is *stable* if for every neighborhood N of any point $p \in X$ there exists a neighborhood M such that $\hat{M} \subset N$ [3].

For $\mathcal{U}$ an open cover of X and $n \geqslant 1$ let $X^n(\mathcal{U}) = \{(x_1, \ldots, x_n) \in X^n: \{x_1, \ldots, x_n\} \subset U \in \mathcal{U}\}$ with the relative topology. Let T^{n-1} denote the standard $(n-1)$ simplex in Euclidean n-space: $T^{n-1} = \{(t_1, \ldots, t_n) \in R^n: t_i > 0, \Sigma t_i = 1\}$.

A *local convex structure* for a space X consists of an open cover $\mathcal{U}$ and a sequence of maps $\lambda^n: X^n(\mathcal{U}) \times T^{n-1} \to X$, $n > 1$, such that

(i) $\lambda^n(x_1, \ldots, x_n; t_1, \ldots, t_n) = \lambda^{n-1}(x_1, \ldots, \bar{x}_m, \ldots, x_n; t_1, \ldots, \bar{t}_m, \ldots, t_n)$ if $t_m = 0$,

(ii) for every neighborhood N of any point $p \in X$ there exists a neighborhood M such that $\lambda^n(M^n \times T^{n-1}) \subset N$ for all n [5].

X is called *stably* LEC if it admits a local equiconnecting function, and X is LCS if it admits a local convex structure.

If such a map λ is defined on the whole $X \times X \times I$ then X is stably EC or CS respectively.

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2. Definition 1. Let a set and a function S be given that satisfy the following conditions:

(1) $S: X \times I \times X \ni (x, t, y) \mapsto S_x(t, y) \in X$,

(2) $S_x(0, y) = y$, $S_x(1, y) = x$ for any $x, y \in X$.

Then for any nonempty set $A \subset X$ let $\operatorname{coS} A = \inf\{D \subset X: A \subset D$ and for any $x \in A, t \in I, S_x(t, D) \subset D\}$. For $A = \varnothing$ let $\operatorname{coS} A = \varnothing$. If $\operatorname{coS} A = A$ then A is S-convex.

The above definition is correct (i.e. the infimum exists) because for any two sets E, D such that, for any $x \in A$ and $t \in I$, $S_x(t, D) \subset D$ and $S_x(t, E) \subset E$ we have $S_x(t, D \cap E) \subset D$ and $S_x(t, D \cap E) \subset E$ which implies $S_x(t, D \cap E) \subset D \cap E$.

Proposition 1. *If $\{A_s\}_{s\in T}$ is a family of S-convex sets, then $\bigcap_{s\in T} A_s$ is S-convex.*

Proof. Suppose that $\bigcap_{s\in T} A_s \neq \varnothing$. For any $x \in \bigcap_{s\in T} A_s$, $t \in I$ and $s \in T$ we have that $S_x(t, \bigcap_{s\in T} A_s) \subset A_s$ and consequently $S_x(t, \bigcap_{s\in T} A_s) \subset \bigcap_{s\in T} A_s$, which means that $\bigcap_{s\in T} A_s$ is S-convex.

Definition 2. A space is S-contractible if S satisfies the conditions (1), (2), and, for any $x \in X$, $\{S_x(t, \cdot)\}$ is a homotopy joining the identity with a constant map (cf. [**1**, p. 22]).

Definition 3. A space X is of C type I if C is a subset of X and there exists S such that X is S-contractible and

(3) for any $x \in C$ and any neighborhood N of x there exists a neighborhood U such that $\operatorname{coS} U \subset N$.

If $C = X$ then we say it is of type I.

Obviously any stably EC is of type I.

Let (M, d) be a metric space. For the nonempty sets $A, D \subset M$ and $r > 0$ let us write $d(A, D) = \inf\{d(x, y): x \in A, y \in D\}$, $B(A, r) = \{x \in M: d(A, x) < r\}$ and $\operatorname{dia} A = \sup\{d(x, y): x, y \in A\}$.

Theorem 1. *Let (M, d) be a metric space and let $A = \bar{A}$ be of ∂A type* I *(∂A denotes the boundary of A) such that, for any $x \notin A$, $d(x, A) = d(x, \partial A)$. Then A is retract of M.*

Proof. Let $\{U_s\}_{s\in T}$ be a locally finite open cover of $M \setminus A$ with a well-ordered family of indices T and, for $\{a_s\}_{s\in T} \subset \partial A$, let the following condition be satisfied: if $x \in U_s$, then $d(x, a_s) \leqslant 2d(x, A)$ for $s \in T$ ([**1**, p. 70]).

For $x \in M \setminus A$ let us consider $T_x := \{s \in T: x \in U_s\}$ and let

$$c_s(x) = d(x, M \setminus U_s)/\sup\{d(x, M \setminus U_s): s \in T_x\} \tag{4}$$

and let

$$r(x) = \begin{cases} x & \text{for } x \in A, \\ S_{a_{s_1}}\Big(c_{s_1}(x), S_{a_{s_2}}\Big(c_{s_2}(x), \dots, \Big(S_{a_{s_n}}\big(c_{t_n}(x), y\big) \dots\Big)\Big)\Big) & \text{for } x \in M \setminus A, \end{cases} \tag{5}$$

where $\{s_1, s_2, \dots, s_n\} = T_x$ and $s_1 < s_2 < \cdots < s_n$ and $y \in A$.

It is easily seen that there always exists $s \in T_x$ such that $c_s(x) = 1$; then $S_{a_s}(c_s(x), z) = a_s$ for $z \in A$. It is trivial that $r(x) \in \mathrm{coS}\{B(x, 2d(x, A)) \cap A\}$. So it follows from (3) that r is continuous on ∂A. Also r is continuous on $M \setminus A$ as for any $x \in M \setminus A$ there exists $B(x, \delta(x))$ which meets only finitely many U_s. Then T_z is finite and fixed for $z \in B(x, \delta(x))$, and r is a finite superposition of the same continuous maps in $B(x, \delta(x))$.

Proposition 2. *Any metric space which is of type* I *is an* $AR(\mathfrak{M})$.

Corollary 1. *Any metrizable space which is of type* I *is* CS (*cf.* [**5**]).

Definition [**1**, p. 219]. A compact space X that is metrizable in such a way that for any $x, y \in X$ there exists exactly one z such that $\rho(x, z) = \rho(y, z) = \rho(x, y)/2$ is called a *strongly convex compactum.*

Corollary 2. *Any strongly convex compactum is* AR (*cf.* [**1**, *p.* 219]).

Definition 4. A space X is of C type II provided that $C \subset X$ and there exists S such that X is S-contractible and the following condition holds:

(6) for any neighborhood N of any $x \in C$ there exists a neighborhood U such that for every $z \in U \cap C$ and $t \in I$ we have $S_z(t, U) \subset N$.

If $C = X$ let us call it type II.

It is easily seen that every type I is type II.

If X is a locally compact space which is S-contractible and S is a map, and if $S_x(t, x) = x$ for all $x \in X$, then X is of type II.

Proposition 3. *Let* $A = \bar{A}$ *be a* ∂A *type* II *subset of a metric space* (M, d) *such that, for any* $x \notin A$, $d(x, A) = d(x, \partial A)$ *and* $M \setminus A$ *is finite dimensional. Then* A *is retract of* M.

Proof. Let $\dim M \setminus A \leqslant n$. Then we may assume that every $x \in M \setminus A$ belongs to at most $n + 1$ sets of $\{U_s\}_{s \in T}$ and we follow the proof of Theorem 1. Condition (6) then ensures the continuity of r on ∂A.

Theorem 2. *Let* $A = \bar{A}$ *be a* ∂A *type* II *subset of a finite dimensional subspace of a linear normed space* $(X, \| \ \|)$. *Then* A *is a retract of* X.

Proof. We construct a dense set E in ∂A in a special way.

1°. Let $E_1 \subset \partial A \cap B(0, 1)$ be a minimal set with respect to the property that for every $x \in \partial A \cap B(0, 1)$, $d(x, E_1) \leqslant 1$. We denote the elements of E_1 by the natural numbers.

2°. We complete E_1 to $E_2 \subset \partial A \cap B(0, 2)$ a minimal set with respect to the property that for any $x \in \partial A \cap B(0, 2)$, $d(x, E_2) < 1/2$ and sign "new" points by the further numbers, etc.

n°. We complete E_{n-1} to $E_n \subset \partial A \cap B(0, n)$; for $x \in \partial A \cap B(0, n)$, $d(x, E_n) < 1/n$.

Now let $E = \bigcup_{n=1}^{\infty} E_n$ and for $a_n \in E$ let

$$c_n(x) = \max\{0, \min\{1, 3 - d(x, a_n)/d(x, A)\}\} \tag{7}$$

and for $x \in X, y \in A$,

$$p_1(x, y) = S_{a_1}(c_1(x), y),$$
$$p_n(x, y) = p_{n-1}\big(x, S_{a_n}(c_n(x), y)\big) \quad \text{for } n > 1. \tag{8}$$

We define $r: X \to A$ as follows:

$$r(x) = \begin{cases} x & \text{for } x \in A, \\ \lim_{n\to\infty} p_n(x, y) & \text{for } x \notin A. \end{cases} \tag{9}$$

The set A is contained in a finite dimensional subspace of X, which with the linearity of norm yields that for $x \in B(0, r)$ each of the sets $A \cap B(x, 2d(x, A))$ and $A \cap B(x, 3d(x, A)) \setminus B(x, 2d(x, A))$ contains at least one and not more than k elements of the sets $E_{n(x)}$, where $n(x) = \max\{[8/d(x, A)], [(r + 1)/6]\}$. In view of the construction of E we need not consider the superposition of more than k maps because there are at most $m \leqslant k$ coefficients $c_n(x) \in (0, 1)$ before the first one that is equal to 1. Therefore r is continuous on $X \setminus A$. The continuity on ∂A follows from (6).

DEFINITION 5. A space X is locally S-contractible if there exists S satisfying (1), (2) and

(10) for any $x \in X$ there exists a neighborhood U such that, for any $z \in U$, $\{S_z(t, \cdot)\}|_U$ is a homotopy joining the identity with a constant map.

DEFINITION 6. A space X is locally C type I (C type II) if it is locally S-contractible and (3) ((6)) is satisfied.

It is obvious that every LEC space is locally S-contractible and every stably LEC space is locally type I.

THEOREM 3. *Let $A = \bar{A}$ be locally A type* I *in a metric space* (M, d) *such that, for every $x \notin A$, $d(x, A) = d(x, \partial A)$. Then A is a retract of D, if D is as follows.*

$$D = \{x \in M\colon \text{there exists } \varepsilon > 0 \text{ such that, for } y, z \in \operatorname{coS}\{B(x, d(x, A) + \varepsilon) \cap \partial A\} \text{ and } z \in \partial A,\ S_z \text{ is a map}\}. \tag{11}$$

PROOF. Let $\varepsilon(x) = \sup\{\varepsilon\colon$ such that for $y, z \in \operatorname{coS}\{B(x, d(x, A) + \varepsilon) \cap \partial A\}$ and $z \in \partial A$, S_z is a map$\}$ and let $(x_n)_{n\in N}$ be any sequence convergent in D, say to x_0. We have $\overline{\lim}_{n\to\infty}\varepsilon(x_n) \leqslant \varepsilon(x_0)$ because otherwise there would exist $n \in N$ and $\delta > 0$ such that $B(x_0, d(x_0, A) + \varepsilon(x_0) + \delta) \subset B(x_n, d(x_n, A) + \varepsilon(x_n))$. Similarly $\varepsilon(x_0) \leqslant \underline{\lim}_{n\to\infty}\varepsilon(x_n)$ because otherwise $B(x_n, d(x_n, A) + \varepsilon(x_n) + \delta) \subset B(x_0, d(x_0, A) + \varepsilon(x_0))$ would hold for a $\delta > 0$. Now let $D_\delta = \operatorname{Int}\{x \in D\colon \varepsilon(x) \geqslant \delta\}$ and $\mathfrak{B} = \{B(x, \lambda(x)) \cap D_{\lambda(x)}\colon x \in D \setminus A\}$ where

$$\lambda(x) = \min\{d(x, A), \varepsilon(x)/4\}. \tag{12}$$

If for $x \in D \setminus A$ there exists $\delta > 0$ such that, for each $y \in B(x, \delta)$, $\lambda(y) \geqslant \lambda(x)$ then $x \in D_{\lambda(x)}$, otherwise a y can be found such that $\lambda(y) < \lambda(x)$ (implies $x \in D_{\lambda(y)}$) and $x \in B(y, \lambda(y))$. Hence $\mathfrak{B}$ is an open cover of $D \setminus A$ and we can find a locally finite open cover $\{U_s\}_{s\in T}$ which is a star refinement of $\mathfrak{B}$. If for $s \in T$, $x_s \in U_s$, we choose z for which $\mathrm{St}(U_s, \mathfrak{U}) \subset B(z, \lambda(z)) \cap D_{\lambda(z)}$ and $a_s \in B(x_s, d(x_s, A) + \lambda(z)) \cap \partial A$, then for $x \in U_s$ we have

$$\begin{aligned} d(x, a_s) &\leqslant d(x, x_s) + d(x_s, a_s) \leqslant \lambda(z) + d(x_s, A) + \lambda(z) \\ &\leqslant 2\lambda(z) + d(x_s, x) + d(x, A) \leqslant 3\lambda(z) + d(x, A) \\ &< d(x, A) + \varepsilon(x). \end{aligned}$$

Now it is easily seen that for these $\{U_s\}_{s\in T}$ and $\{a_s\}_{s\in T}$ formulas (4) and (5) give the required retraction of D.

Proposition 4. *Any metric space which is locally type* I *is an ANR* ($\mathfrak{M}$).

Proof. In the previous considerations we put everywhere "A" in place of "∂A". We see that $D' = \bigcup_{\delta>0} D'_\delta$ so obtained is open. If $x \in A$ then there is $\delta > 0$ for which $x \in D'$ and hence $A \subset D'$.

Corollary. *Any metrizable locally type* I *space is LCS* (*cf.* [5]).

Proposition 5. *Let* $A = \bar{A}$ *be locally* A *type* I *in a metric space* (M, d) *such that, for every* $x \notin A$, $d(x, A) = d(x, \partial A)$ *and* $\inf\{\sup\{r\colon S_z \text{ is a map for } z, y \in \mathrm{coS}\{B(x, r) \cap \partial A\} \text{ and } z \in \partial A\}\colon x \in \partial A\} = a > 0$. *Then* A *is a retract of* $B(A, a/2)$.

Proof. It is enough to show that $B(A, a/2) \subset D$, where D is defined by (11). Let $x \in B(A, a/2)$. Then $\delta(x) := a/2 - d(x, A) > 0$ and

$$\begin{aligned} \mathrm{dia}\{B(x, d(x, A) + \delta(x)/2) \cap A\} &\leqslant 2(d(x, A) + \delta(x)/2) \\ &= 2(d(x, A) + a/4 - d(x, A)/2) = d(x, A) + a/2 < a. \end{aligned}$$

Theorem 4. *Let* $A = \bar{A}$ *be a compact type* I *subset of a metric space* (M, d) *and let* $f\colon A \to M$ *be a map. For each* $x \in A$ *and* $\varepsilon > 0$ *let* $A(f(x), \varepsilon) = \overline{\mathrm{coS}}\{B(f(x), d(f(x), A) + \varepsilon) \cap A\}$. *Then there is an* $x \in A$ *such that* $x \in \bigcup_{\varepsilon>0} A(f(x), \varepsilon)$ (*this latter set will be denoted by* $A_{f(x)}$).

Proof. Suppose that there exists $\delta > 0$ such that, for all $x \in A$, $x \notin A(f(x), \delta)$. Then we take δ in place of $\varepsilon'(x)$ and repeat the construction of r from Proposition 4. The map $r \circ f\colon A \to A$ has a fixed point [1, p. 101] which is impossible as $(r \circ f)(x) \in A(f(x), \delta)$. Hence there exists a sequence $(x_n)_{n\in N}$ such that $x_n \in A(f(x_n), \delta_n)$ with $\delta_n \to 0$; we may assume the sequence to converge, say to x. For any $\delta > 0$ there exists n_0 such that for every $n \geqslant n_0$

$$B(f(x_n), d(f(x_n), A) + \delta_n) \subset B(f(x), d(f(x), A) + \delta).$$

Therefore $x_n \in A(f(x), \delta)$ for $n \geqslant n_0$ and $x \in A(f(x), \delta)$. So it must be that $x \in A_{f(x)}$.

Theorem 3 and Proposition 5 have locally type I analogs; the assumption that, for $x \notin A$, $d(x, A) = d(x, \partial A)$ can be omitted.

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