HOW TO RECOGNIZE A LOCALIZED SPHERE

J. AGUADÉ

Abstract. Two cohomological characterizations of the sphere localized at a prime are given.

The purpose of this note is to obtain necessary and sufficient conditions on the cohomology of a nilpotent space in order to insure that it has the homotopy type of \( S^m_p \), the sphere localized at the prime \( p \) (cf. [2]). \( p \) will denote a fixed prime integer. \( \hat{\mathbb{Z}}_p, \mathbb{Z}_{(p)}, \mathbb{Z}_n, \mathbb{Z}_{p^m} \) denote the \( p \)-adic integers, the integers localized at \( p \), the integers modulo \( n \) and the \( p^m \)th roots of unity with \( m \) running over all integers \( > 0 \), respectively. Group will always mean abelian group. A group is called \( p \)-local if it is \( q \)-divisible for every \( q \neq p \) and has no \( q \)-torsion for every \( q \neq p \). A nilpotent space is called \( p \)-local if its homology groups with integral coefficients are \( p \)-local [2, p. 53].

Recall that a subgroup \( B \) of a group \( A \) is called \( p \)-basic if the following holds:

(i) \( B \) is a direct sum of cyclic \( p \)-groups and infinite cyclic groups,
(ii) \( A/B \) is \( p \)-divisible,
(iii) \( B/p^kB \) is a direct summand of \( A/p^kB \) for all \( k \).

Every group contains \( p \)-basic subgroups [1, 1, p. 137].

We use freely the most elementary properties of the functors Hom and Ext. A standard reference is [1, Chapters VIII and IX]. In particular we shall use the following:

\[
\text{Ext}(\mathbb{Z}_n, A) = A/nA, \quad \text{Ext}(\mathbb{Z}_{p^m}, \mathbb{Z}_p) = \mathbb{Z}_p, \quad \text{Ext}(\mathbb{Q}, \mathbb{Z}_{(p)}) = \mathbb{Q}^{\times_0}, \quad \text{Ext}(\mathbb{Z}_p^m, \mathbb{Z}_{(p)}) = \hat{\mathbb{Z}}_p.
\]

All spaces are assumed to be of the homotopy type of CW-complexes.

Lemma 1. Let \( B \) be a \( p \)-basic subgroup of \( A \). Then \( \text{Hom}(A, \mathbb{Z}_p) = \text{Hom}(B, \mathbb{Z}_p) \).

Proof. Since \( A/B \) is \( p \)-divisible and \( \mathbb{Z}_p \) contains no \( p \)-divisible subgroups other than 0, the Hom-Ext exact sequence associated to \( \mathbb{B} \to A \to A/B \) yields that \( \text{Hom}(A, \mathbb{Z}_p) \) is a subgroup of \( \text{Hom}(B, \mathbb{Z}_p) \). Let \( f: B \to \mathbb{Z}_p \). Clearly, \( f \) factorizes through \( f': B/pB \to \mathbb{Z}_p \). Since \( B \) is a \( p \)-basic subgroup of \( A \), \( B/pB \) is a direct summand of \( A/pB \). Hence \( f' \) extends to \( A/pB \) and yields a homomorphism \( g \in \text{Hom}(A, \mathbb{Z}_p) \).

Lemma 2. If \( A \) is a \( p \)-local group such that \( \text{Hom}(A, \mathbb{Q}) = \text{Hom}(A, \mathbb{Z}_p) = 0 \), then \( A \) is a divisible \( p \)-group.
Proof. Since \( \text{Hom}(A, \mathbb{Q}) = 0 \), \( A \) has no elements of infinite order. Since \( A \) is \( p \)-local, \( A \) is a \( p \)-group. Let \( B \) be a \( p \)-basic subgroup. By Lemma 1, \( \text{Hom}(B, \mathbb{Z}_p) = \text{Hom}(A, \mathbb{Z}_p) = 0 \) and since \( B \) is a direct sum of cyclic \( p \)-groups then \( B = 0 \). That is \( A \) is \( p \)-divisible and so, since \( A \) is \( p \)-local, \( A \) is divisible. □

**Lemma 3.** Let \( E \) be a nilpotent space such that \( H_*(E; \mathbb{Z}) = H_*(S^m_p; \mathbb{Z}) \). Then \( E \) has the homotopy type of \( S^m_p \).

Proof. If \( m > 1 \) then \( H_1E = 0 \). In other words, the abelianization of \( \pi_1E \) is trivial. Since \( \pi_1E \) is a nilpotent group we conclude that \( E \) is simply connected and so \( E \) is a Moore space \( M(\mathbb{Z}(p), m) \), that is a localized sphere \( S^m_p \).

If \( m = 1 \) then we have

\[
\left[ E, S^1_p \right] = \left[ E, K(\mathbb{Z}(p), 1) \right] 
\cong H^1(E; \mathbb{Z}(p)) = \mathbb{Z}(p)
\]

where \( K(\mathbb{Z}(p), 1) \) denotes an Eilenberg-Mac Lane space of type \((\mathbb{Z}(p), 1)\). Then there exists a map \( f: E \rightarrow S^1_p \) which induces isomorphisms in homology. Because \( E \) and \( S^1_p \) are nilpotent spaces, \( f \) must be a homotopy equivalence. The result follows. □

**Theorem 4.** Let \( E \) be a nilpotent space such that

1. \( E \) is \( p \)-local,
2. \( H^*(E; R) = H^*(S^m_p; R) \) where \( R = \mathbb{Z}_p \) or \( R = \hat{\mathbb{Z}}_p \),
3. \( H^*(E; Q) = H^*(S^m_p; Q) \),
4. \( H^{m+1}(E; Z(p)) = 0 \).

Then \( E = S^m_p \). If any of the above conditions is omitted then the conclusion is false.

Proof. It suffices to prove that \( H_*(E; \mathbb{Z}) = H_*(S^m_p; \mathbb{Z}) \). Let us consider the exact sequence \( \hat{\mathbb{Z}}_p \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p \). It induces a long exact sequence in cohomology:

\[
\ldots \rightarrow H^r(E; \hat{\mathbb{Z}}_p) \rightarrow H^r(E; \mathbb{Z}_p) \rightarrow H^r(E; \mathbb{Z}_p) \rightarrow H^{r+1}(E; \hat{\mathbb{Z}}_p) \rightarrow \ldots
\]

From this it follows easily that we may assume, without loss of generality that \( R = \mathbb{Z}_p \) in condition (2). On the other hand, condition (3) implies that \( H_rE \) has torsion-free rank zero if \( r \neq m \) and has torsion-free rank one if \( r = m \).

The universal coefficient theorem:

\[
\text{Ext}(H_{r-1}E, \mathbb{Z}_p) \rightarrow H^r(E; \mathbb{Z}_p) \rightarrow \text{Hom}(H_rE, \mathbb{Z}_p)
\]

yields: \( \text{Hom}(H_rE, \mathbb{Z}_p) = 0 \) if \( r \neq m \) and \( \text{Ext}(H_rE, \mathbb{Z}_p) = 0 \) if \( r \neq m - 1 \). It follows from Lemma 2 that \( H_rE \) is a divisible \( p \)-group if \( r \neq m \). The structure theorem for divisible groups [1, I, p. 104] shows that \( H_rE = \bigoplus \mathbb{Z}_p^m \). But \( \text{Ext}(H_rE, \mathbb{Z}_p) = 0 \) if \( r \neq m - 1 \), hence \( H_rE = 0 \) if \( r \neq m, m - 1 \) because \( \text{Ext}(\mathbb{Z}_p^m, \mathbb{Z}_p) = \mathbb{Z}_p \).

Since \( H^{m+1}(E; \mathbb{Z}_p) = \mathbb{Z}_p \) the universal coefficient theorem shows that there are two cases:

(A) \( \text{Ext}(H_{m-1}E, \mathbb{Z}_p) = \mathbb{Z}_p \), \( \text{Hom}(H_mE, \mathbb{Z}_p) = 0 \).

\( H_{m-1}E \) is a divisible \( p \)-group and so \( H_{m-1}E = \bigoplus \mathbb{Z}_p^m \). Since \( \text{Ext}(H_{m-1}E, \mathbb{Z}_p) = \mathbb{Z}_p \) it follows that \( H_{m-1}E = \mathbb{Z}_p^m \). On the other hand, let \( B \) be a \( p \)-basic subgroup of \( H_mE \). By Lemma 1, \( \text{Hom}(B, \mathbb{Z}_p) = \text{Hom}(H_mE, \mathbb{Z}_p) = 0 \) and so \( B = 0 \) and \( H_mE \) is divisible because it is \( p \)-local and \( p \)-divisible. Since \( H_mE \) has torsion-free rank one and \( \text{Ext}(H_mE, \mathbb{Z}_p) = 0 \) we conclude that \( H_mE = \mathbb{Q} \). Hence, we see that in this case
the space $E$ has the homology of $M(Q, m) \vee M(Z_p^m, m - 1)$, the wedge of the Moore spaces $M(Q, m)$ and $M(Z_p^m, m - 1)$.

(B) $\text{Ext}(H_{m-1}E, Z_p) = 0$, $\text{Hom}(H_{m}E, Z_p) = Z_p$.

$H_{m-1}E$ is a divisible $p$-group such that $\text{Ext}(H_{m-1}E, Z_p) = 0$. Since $\text{Ext}(Z_p^m, Z_p) = Z_p$ then $H_{m-1}E = 0$ and so $E$ has homology $\neq 0$ only in dimension $m$. Let $B$ be a $p$-basic subgroup of $H_mE$. We have $\text{Hom}(B, Z_p) = \text{Hom}(H_mE, Z_p) = Z_p$. Since $B$ is a direct sum of infinite cyclic groups and cyclic $p$-groups, either $B = Z$ or $B = Z_p^\alpha$. If $B = Z_p^\alpha$ then condition (iii) in the definition of $p$-basic subgroup yields that $Z_p^\alpha$ is a direct summand of $H_mE$, but this is impossible because $\text{Ext}(H_mE, Z_p) = 0$. Hence $Z$ is a $p$-basic subgroup of $H_mE$. Since $D = H_mE/Z$ is divisible and $\text{Ext}(H_mE, Z_p) = 0$, the Hom-Ext exact sequence associated to $Z \rightarrow H_mE \rightarrow D$ gives $\text{Ext}(D, Z_p) = 0$. If we apply now the structure theorem for divisible groups we obtain that $D$ has no $p$-torsion and so $H_mE$ is torsion-free because it is $p$-local and so it can only have $p$-torsion. Then $H_mE$ is a $p$-local torsion-free group of rank one. We apply now the classification theorem for these groups [1, II, p. 110]. Since $H_mE$ is $p$-local, in order to prove that $H_mE = Z(p)$ it suffices to show that $H_mE$ contains elements of $p$-height zero. Let us consider $1 \in Z \subset H_mE$. If $1 = pa$, $a \in H_mE$ then in $D$ we have $0 = p\bar{a}$ and since $D$ has no $p$-torsion we get $a \in Z$, a contradiction. Hence, the type of $H_mE$ is $t_q = \infty$ if $q \neq p$ and $t_p = 0$. Then the space $E$ has the same integral homology as $S^m$ and then, by Lemma 3, $E = S^m$.

We have proved that if a nilpotent space $E$ satisfies conditions (1), (2), (3) of the theorem then either $E = S^m$ or $E$ has the same homology as $M(Q, m) \vee M(Z_p^m, m - 1)$. It suffices to show that if $H_mE = Q$ then $E$ does not satisfy condition (4). This follows easily from the fact $\text{Ext}(Q, Z(p)) = Q^{\oplus \infty}$.

Finally, note that none of the conditions (1), (2), (3), (4) can be omitted. Let us consider the spaces $E_1 = S^m \vee M(Z_q, 2)$ ($q$ a prime $\neq p$); $E_2 = S^m \vee M(Z_p, 2m)$; $E_3 = M(Z_p^m, m - 1)$; $E_4 = M(Q, m) \vee M(Z_p^m, m - 1)$. It is easily seen that the space $E_i$ verifies conditions (1), (2), (3), (4) except the $i$th, but $E_i \neq S^m$.

The above theorem shows that the cohomology with coefficients $Z_p$, $Z_p^\alpha$, $Q$ is not enough to characterize the localized spheres. The following theorem shows that the cohomology with coefficients $Z(p)$ is suitable for this purpose.

**THEOREM 5.** Let $E$ be a $p$-local nilpotent space such that $H^*(E; Z(p)) = H^*(S^m; Z(p))$. Then $E = S^m$.

**PROOF.** From the universal coefficient theorem we obtain $\text{Hom}(H,E, Z(p)) = 0$ if $r \neq m$ and $\text{Ext}(H,E, Z(p)) = 0$ if $r \neq m - 1$. Further, $\text{Ext}(H_{m-1}E, Z(p))$ is a $p$-local subgroup of $Z(p)$. Then either $\text{Ext}(H_{m-1}E, Z(p)) = Z(p)$ or $\text{Ext}(H_{m-1}E, Z(p)) = 0$. But in the first case $Z(p)$ will be a cotorsion group (cf. [1, I, p. 232] and Theorem 54.6 of [1, I, p. 235]) and this is impossible because $\text{Ext}(Q, Z(p)) = Q^{\oplus \infty} \neq 0$. Hence, $\text{Ext}(H,E, Z(p)) = 0$ for all $r$.

Let $B_r$ be a $p$-basic subgroup of $H,E$. If $B_r$ contains a direct summand of the form $Z_p^\alpha$, then it follows easily from the definition of $p$-basic subgroup that $Z_p^\alpha$ is a direct summand of $H,E$. But $\text{Ext}(Z_p^\alpha, Z(p)) = Z(p)/p^kZ(p) \neq 0$. Hence, $B_r$ is free. Let us consider the Hom-Ext exact sequence associated to $Z(p) \rightarrow Z(p) \rightarrow Z_p$:
0 → Hom(H_rE, \mathbb{Z}(p)) \rightarrow \mathbb{Z}(p) → Hom(H_rE, \mathbb{Z}_p) → 0.

We get that Hom(H_rE, \mathbb{Z}_p) = 0 if r \neq m. By Lemma 1, Hom(B_r, \mathbb{Z}_p) = Hom(H_rE, \mathbb{Z}_p) = 0, hence B_r = 0 if r \neq m. This shows that H_rE is a divisible group for r \neq m. But the structure theorem for divisible groups and the equalities Ext(Q, \mathbb{Z}(p)) = Q^\mathbb{N}, Ext(\mathbb{Z}_p, \mathbb{Z}(p)) = \hat{\mathbb{Z}}_p, Ext(H_rE, \mathbb{Z}(p)) = 0 yield H_rE = 0 if r \neq m.

In the case r = m the above exact sequence shows that Hom(H_mE, \mathbb{Z}_p) = \mathbb{Z}_p. By Lemma 1, B_m = \mathbb{Z}. The Hom-Ext exact sequence associated to \mathbb{Z} \rightarrow H_mE \rightarrow H_mE/\mathbb{Z} shows that Ext(H_mE/\mathbb{Z}, \mathbb{Z}(p)) is the image of the countable group Hom(\mathbb{Z}, \mathbb{Z}(p)). Since Ext(Q, \mathbb{Z}(p)) and Ext(\mathbb{Z}_p, \mathbb{Z}(p)) are both uncountable and since H_mE/\mathbb{Z} is a divisible group, we get that H_mE/\mathbb{Z} is a torsion group without p-torsion. This leads to Hom(H_mE/\mathbb{Z}, Q) = 0 and so Hom(H_mE, Q) = Hom(\mathbb{Z}, Q) = Q. Hence H_mE is a group of torsion-free rank one. Further, H_mE is torsion-free because H_mE/\mathbb{Z} has no p-torsion and H_mE is p-local. Now we can show, as in the proof of Theorem 4 that H_mE = \mathbb{Z}(p) and so the space E has the same integral homology as S_p^m. Then, by Lemma 3, E is a localized sphere S_p^m. □

REFERENCES


SECCIÓ DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, BELATTERA (BARCELONA), SPAIN

Current address: Forschungs Institut für Mathematik, ETH-Zentrum, 8092-Zürich, Switzerland