PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, April 1980

AN EXAMPLE CONCERNING INVERSE LIMIT SEQUENCES OF NORMAL SPACES

M. G. CHARALAMBOUS

ABSTRACT. Using techniques developed by Wage and Przymusiński, we construct an inverse limit sequence (X_n, f_{nm}) with limit space X such that each X_n is Lindelöf with $\dim X_n = 0$, where dim denotes covering dimension, while X is normal with $\dim X > 0$. The space X is a counterexample to several conjectures in Topology.

Let X be the inverse limit of the sequence (X_n, f_{nm}). We recall that if, for some k in N, the set of positive integers, $\dim X_n \leq k$ for each n in N, then $\dim X \leq k$ if one of the following conditions is satisfied: (a) X_n is perfectly normal for each n in N, (b) X is strongly paracompact, and (c) X is countably paracompact, X_n is normal and f_{nm} is open and surjective for all n, m in N [1], [4].

THE CONSTRUCTION. Let d be the usual metric on the unit interval I. Let ρ be the separable metric on I introduced by Wage [8]. The ρ-topology on I is finer than the d-topology, $d \leq \rho$ and every ρ-Borel set of I is also d-Borel. Moreover, every nonempty ρ-open set is uncountable and there is an ρ-closed set E such that the boundary of every nonempty ρ-open set disjoint from E has cardinality c, the cardinality of the continuum.

Let $\{A_1, A_2, \cdots\}$ be a partition of I with the property that for every uncountable Borel set of B of I and every n in N, $|B \cap A_n| = c$ [6, Theorem 2]. We may clearly assume that 0, 1 are in A_1 and that $A_1 \cap E$ is ρ-dense in E. For each x in $I - A_1$ and each n in N, choose x_n^-, x_n^+ in A_2 so that

$$x - \frac{1}{n} < x_n^- < x_{n+1}^- < x < x_{n+1}^+ < x_n^+ < x + \frac{1}{n}.$$

Let $\{(A_\alpha, B_\alpha): \alpha < \omega(c)\}$ be the collection of all pairs of countable subsets of A_1^2 with $|\bar{A}_\alpha^\rho \cap \bar{B}_\alpha^\rho \cap \Delta|$ uncountable, where $\omega(c)$ is the first ordinal of cardinality c and Δ is the diagonal of I^2. For each $\alpha < \omega(c)$ and each n in N, choose x_α in A_1, $(x_{\alpha n}^1, x_{\alpha n}^2)$ in A_α and $(x_{\alpha n}^3, x_{\alpha n}^4)$ in B_α so that $\rho(x_{\alpha n}^i, x_\alpha) \leq 1/n$, $x_{\alpha n}^i \lhd x_\alpha$ and $x_\beta \lhd x_\alpha$ for $\beta < \alpha$ and $i = 1, 2, 3, 4$, where $\lhd$ is a well-ordering on I.

For n, m in N and x in I, we define a set $B_m^n(x)$ containing x as follows. For x in $A_2 \cup \cdots \cup A_{n+1} \cup (A_1 - \{x_\alpha: \alpha < \omega(c)\})$, $B_m^n(x) = \{x\}$. For x in $\bigcup_{k=n+2}^\infty A_k$, $B_m^n(x) = [x_m^-, x_m^+]$. On $\{x_\alpha: \alpha < \omega(c)\}$, B_m^n is defined by transfinite induction.

Assuming it has been defined for all $\beta < \alpha$, we set

$$B_m^n(x_\alpha) = \{x_\alpha\} \cup \left(B_{k+2m}^n(x_{\alpha k}^i): k > 2m, i = 1, 2, 3, 4\right).$$

Received by the editors July 6, 1978 and, in revised form, March 2, 1979.
AMS (MOS) subject classifications (1970). Primary 54G20, 54F45.
Key words and phrases. Normal, Lindelöf, paracompact, topologically complete and N-compact space, covering dimension.

It is readily seen that $B^n_m(x)$ is a d-closed subset of $[x - 1/m, x + 1/m]$ and $B^n_m(x_\alpha)$ is a countable subset of $\{x\colon x \lhd x_\alpha, \rho(x, x_\alpha) < 1/m\}$. Also $B^n_{m+1}(x) \subset B^n_m(x)$ and if y is in $B^n_m(x)$, then $B^n_k(y) \subset B^n_m(x)$ for some k in N. It follows that $\{B^n_m(x)\colon m \in N\}$ is a local base at x consisting of open-and-closed sets relative to some topology τ_n on I. Next, put $B_m(x) = B^n_m(x)$ if $x = x_\alpha$ for some $\alpha < \omega(c)$, and $B_m(x) = \{x\}$ if not. $B_m(x)$ is a local base of x consisting of open-and-closed sets relative to some topology τ on I. Clearly, τ is finer than the ρ-topology on I, and $\{x^i_{\alpha n}\colon n = 1, 2, \dots\}$ converges to x_α, $i = 1, 2, 3, 4$.

In the sequel, X, X_n denote (I, τ), (I, τ_n), respectively, and $f_{nm}\colon X_m \to X_n$ denotes the identity function. It is readily verified that X is the limit space of the inverse limit sequence (X_n, f_{nm}).

CLAIM 1. *X_n is a T_2 Lindelöf space with* $\dim X_n = 0$.

PROOF. It is obvious that X_n is T_2 and $\operatorname{ind} X_n = 0$. Let $\mathcal{U}$ be an open cover of X_n. Since clearly every point of A_{n+2} has a local base consisting of open intervals, there are d-open sets $G_1, G_2, \dots,$ each contained in some member of $\mathcal{U}$, such that $A_{n+2} \subset G = \bigcup_{i=1}^{\infty} G_i$. Then, since $(X - G) \cap A_{n+2} = \varnothing$, $X - G$ is countable. It is now clear that $\mathcal{U}$ has a countable open refinement, and hence X_n is Lindelöf. Thus, since also $\operatorname{ind} X_n = 0$, $\dim X_n = 0$.

CLAIM 2. *The family of neighbourhoods of the diagonal of X^2 is a uniformity. Hence X is normal* [2, *Problem* L, *p.* 209].

PROOF. Let G be an open neighbourhood of Δ. It suffices to find a neighbourhood V of Δ with $V \circ V \subset G$.

Let $A = A_1^2 - G$. If $\bar{A}^\rho \cap \Delta$ were uncountable, then, for some $\alpha < \omega(c)$, A would contain $A_\alpha = B_\alpha$ as a countable ρ-dense subset and hence (x_α, x_α) would be a point of A. Thus, $\bar{A}^\rho \cap \Delta$ is countable. Since ρ is separable and X^2 is Tychonoff, there is a cozero cover $\{G_1, G_2, \dots\}$ of the open-and-closed subset A_1 of X such that

$$\Delta \cap A_1^2 \subset \bigcup_{i=1}^{\infty} G_i^2 \subset G.$$

Let $\{H_1, H_2, \dots\}$ be a cozero star-refinement of $\{G_1, G_2, \dots\}$ and put $V = \bigcup_{i=1}^{\infty} H_i^2 \cup \Delta$. It is readily verfied that V has the required properties.

CLAIM 3. $\dim X > 0$.

PROOF. Let F be an uncountable ρ-closed set of I with $E \cap F = \varnothing$, and suppose U, V are disjoint open-and-closed sets of A_1 with $E \cap A_1 \subset U$, $F \cap A_1 \subset V$ and $A_1 = U \cup V$. Then V is uncountable and $\bar{U}^\rho \cap \bar{V}^\rho$ is countable, otherwise, for some $\alpha < \omega(c)$, A_α, B_α would be countable ρ-dense subsets of U^2, V^2, respectively, and hence x_α would be in $U \cap V$. Also $\bar{U}^\rho \cup \bar{V}^\rho = \bar{A}_1^\rho = X$ and, since $A_1 \cap E$ is ρ-dense in E, $E \subset \bar{U}^\rho$. Hence $X - \bar{U}^\rho = \bar{V}^\rho - \bar{U}^\rho$ is an uncountable ρ-open set of I contained in $X - E$ with ρ-boundary contained in the countable set $\bar{U}^\rho \cap \bar{V}^\rho$. We conclude that $\dim A_1 > 0$ and hence $\dim X > 0$.

REMARK 1. A space is called N-compact if it is the inverse limit of countable discrete spaces. A zero-dimensional T_2 Lindelöf space is N-compact, and so is the inverse limit of N-compact spaces. Hence X is N-compact although $\dim X > 0$.

Przymusiński's space in [7] has the same property. Spaces exhibiting the same pathology were previously constructed in [3] and [5].

REMARK 2. Kelley [2, p. 209] has conjectured that a topologically complete space with the property that the family of all neighbourhoods of its diagonal is a uniformity is paracompact. X is a counterexample to this conjecture. For if X were paracompact, since it is also locally countable, we would have $\dim X = \text{loc} \dim X = 0$.

REFERENCES

1. M. G. Charalambous, *The dimension of inverse limits*, Proc. Amer. Math. Soc. **58** (1976), 289–295.

2. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1955.

3. S. Mrowka, *Recent results on E-compact spaces*, Proceedings of the Second Pittsburgh International Conference, Lecture Notes in Math., vol. 378, Springer-Verlag, Berlin and New York, 1974, pp. 298–301.

4. K. Nagami, *Countable paracompactness of inverse limits and products*, Fund. Math. **73** (1971), 261–270.

5. E. Pol and R. Pol, *A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an N-compact space of positive dimension*, Fund. Math. **97** (1977), 43–50.

6. T. Przymusiński, *On the notion of n-cardinality*, Proc. Amer. Math. Soc. **69** (1978), 333–338.

7. ______, *On the dimension of product spaces and an example of M. Wage*, Proc. Amer. Math. Soc. **76** (1979), 315–321.

8. M. Wage, *The dimension of product spaces*, Proc. Nat. Acad. Sci. U.S.A. (to appear).

DEPARTMENT OF MATHEMATICS, AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA

Current address: Department of Mathematics, University of Nairobi, Nairobi, P. O. B.30197, Kenya