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PLANE CONTINUA AND TRANSFORMATION GROUPS

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ABSTRACT. If G is a compact transformation group acting on a nonseparating plane continuum, does G have a fixed point? This paper provides some partial answers to the question. In particular, it yields the corollary that a period homeomorphism on a nonseparating plane continuum has a fixed point.

This paper puts together certain techniques in transformation group theory which are useful for certain fixed point theorems in point set topology. It is the author's hope that this paper stimulates some interest in using group theory to deal with topological problems. The proofs of certain theorems might seem painfully detailed to those who are familiar with transformation groups.

A nonseparating plane continuum is a compact connected space whose complement in the plane is connected. Cohomology is the reduced Alexander-Wallace or Čech type and $H^n(X, A)$ is the cohomology of X over the abelian group A. We need first a list of definitions.

(1) (G, M) is a compact transformation group (or simply G acts on M) if G is a compact group so that the action (g, x) into gx is continuous and $g(hx) = (gh)x$ for all g, h in G and x in M.

(2) p is a fixed point for G if $gp = p$ for all g in G. The set of fixed points will be denoted by $F(G, M)$.

(3) A group G is solvable if it has closed subgroups $G = G_1 \supset G_2 \supset \cdots \supset G_n = (1)$ so that G_{i+1} is normal in G_i and G_i/G_{i+1} is abelian.

(4) A group G has property (p) if G/G_0 is solvable and each factor group in the series does not have the p-adic group as a factor (see [HM]). (G_0 denotes the identity component of G.)

(5) A space is acyclic over A if all the reduced cohomology groups over A are zero.

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THEOREM 1. *Let M be a plane continuum. The following statements are equivalent*:
(a) *M does not separate the plane.*
(b) *$H^1(M, Z) = 0$ where Z denotes the integers.*
(c) *$H^1(M, A) = 0$ for some nontrivial abelian group A.*
(d) *$H^1(M, A) = 0$ for any abelian group A.*

PROOF. To prove (a) implies (d), since M does not separate the plane, then M is an inverse limit of 2-cells. Hence by the Continuity Axiom (see [**Sp**, p. 358]), M is acyclic for any abelian group A. (d) implying (c) is obvious.

To prove (c) implies (b), suppose $H^1(M, Z) \neq 0$. Then $H^1(M, Z)$ is a direct sum of the additive integers by the Alexander Duality [**Hu**, p. 138]. By the Universal Coefficient Theorem:

$$0 \to H^1(M, Z) \otimes A \to H^1(M, A) \to H^2(M, Z) * A \to 0$$

is an exact sequence. Since $H^2(M, Z) = 0$, then $H^1(M, A) \neq 0$ due to the fact that tensor product commutes with direct sum. The Alexander Duality gives us that (b) implies (a).

LEMMA. *If G is a compact connected group acting on a nonseparating plane continuum M, then $F(G, M)$ is nonempty and acyclic.*

PROOF. Consider an orbit Gx. Then Gx is at most one-dimensional since a 2-dimensional compact subset of the plane contains a 2-cell and hence cannot be homogeneous. So by Lemma 2, [**Ho**, p. 366], $F(G, M)$ is acyclic over the rationals. Thus $F(G, M)$ is nonempty and does not separate the plane by Theorem 1.

THEOREM 2. *If G is a group with property (p) acting on a nonseparating plane continuum, then G has its set of fixed points a nonseparating continuum.*

PROOF. If G_0 is the identity component of G, then $K = F(G_0, M)$ is a nonseparating plane continuum. Consider $H = G/G_0$ and H acts on K by $(gG_0)x = gx$. Then $H = H_1 \supset H_2 \supset \cdots \supset H_n = (1)$ satisfying the definition of property (p). Let $F_{n-1} = F(H_{n-1}, K)$ which is acyclic. Then H_{n-2}/H_{n-1} acts on F_{n-1} and $F(H_{n-2}/H_{n-1}, F_{n-1})$ is acyclic by Theorem 3.23 of [**HM**, p. 333]. But $F(H_{n-2}, K) = F(H_{n-2}/H_{n-1}, F_{n-1})$. An induction proof shows that $F(H, K)$ is acyclic. Since $F(H, K) = F(G, M)$, the fixed point set is acyclic.

REMARK 1. A finite group acting on an n-cell does not necessarily have a fixed point [**FR**] but a compact group acting on the 2-cell always has a fixed point.

REMARK 2. A periodic homeomorphism on a nonseparating plane continuum has a fixed point since the iterates of the homeomorphism form a finite abelian group.

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