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A NOTE ON NORMAL COMPLEMENTS IN mod p ENVELOPES

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ABSTRACT. Let G be a finite p-group and let $Z_p[G]$ denote the group ring of G over the field of p elements. The mod p envelope of G, denoted by G^*, is the set of elements of $Z_p[G]$ with coefficient-sum equal to 1. Many examples of p-groups that have a normal complement in G^* have been found, including ten of the fourteen different groups of order 16. This note proves that one of the remaining groups of order 16 has a normal complement. The remaining groups of order 16 are the dihedral, semidihedral, and generalized quaternion groups of order 2^n, $n = 4$. We will prove that these groups do not have a normal complement for any $n > 4$.

When G is a finite p-group and Z_p is the field of integers modulo p, it is well known that the group ring $Z_p[G]$ is a local ring with units $U(Z_p[G]) = \{\Sigma k_g g \in Z_p[G]: \Sigma k_g \neq 0\}$. The mod p envelope of G is $G^* = \{\Sigma k_g g: \Sigma k_g = 1\}$, and it is easy to see that G^* is a finite p-group. D. L. Johnson [1] defined the class L_p of finite p-groups to consist of those G which have a normal complement in G^*. He made several observations about this class of groups and gave numerous examples of groups belonging to this class. He pointed out that the "smallest" group not definitely known to belong to L_2 is the dihedral group of order 16. This group is not in L_2 and, in fact,

THEOREM 1. *The dihedral, semidihedral, and generalized quaternion 2-groups of order* 16 *and greater are not in* L_2.

PROOF. Let G be the dihedral group $\langle a, b: a^2 = 1, b^{2^{n+1}} = 1, a^{-1}ba = b^{-1}\rangle$, or the semidihedral group $\langle a, b: a^2 = 1, b^{2^{n+1}} = 1, a^{-1}ba = b^{2^n-1}\rangle$ or the generalized quaternion group $\langle a, b: a^2 = b^{2^n}, b^{2^{n+1}} = 1, a^{-1}ba = b^{-1}\rangle$, where $n \geq 2$.

First consider the case $n = 2$, i.e, G has order 16. Let $N \lhd G^*$ and $G^* = GN$. We will show that $b^4 \in N$, and hence N cannot be a normal complement. For any $g \in G$, either $g^2 \in \langle b^4\rangle$ or $(bg)^2 \in \langle b^4\rangle$. Since $N \lhd G^*$ and $G^* = GN$, for any $\alpha \in G^*$, either $\alpha^2 \in \langle b^4\rangle N$ or $(b\alpha)^2 \in \langle b^4\rangle N$. Let $(u, v) = u^{-1}v^{-1}uv$ denote the commutator of u and v. Clearly $(\langle b^4\rangle N, G^*) \subseteq N$, and thus for any $\alpha \in G^*$, either $(\alpha^2, \beta) \in N$ for all $\beta \in G^*$, or $((b\alpha)^2, \beta) \in N$ for all $\beta \in G^*$.

Note that $a^{-1}ba = b^{4j-1}$ where $j = 1$ when G is the semidihedral group and $j = 0$, otherwise. Let $\alpha = 1 + a + b \in G^*$. It is easy to compute that

$$\alpha^2 = 1 + a^2 + b^2 + a(b + b^{4j-1})$$

and

$$\begin{aligned}(\alpha^2)^{ab^{-2}} &= 1 + a^2 + b^{-2} + a(b^{4j-1} + b)b^{-4}\\ &= \left[b^4 + a^2b^4 + b^2 + a(b^{4j-1} + b)\right]b^4 = \alpha^2b^4,\end{aligned}$$

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since $a^2 = 1$ or b^4. Thus $(\alpha^2, ab^{-2}) = b^4$. It is also easy to compute that

$$(b\alpha)^2 = b^2 + b^4 + a^2b^{4j} + a(b^{-2} + b^{4j} + b^{4j+1} + b^{4j-3})$$

and

$$\begin{aligned}\left[(b\alpha)^2\right]^{ab^{-3}} &= b^{-2} + b^4 + a^2b^{4j} + a(b^2 + b^{4j} + b^{-1} + b^3)b^{4j-6}\\ &= \left[b^2 + 1 + a^2b^{4j-4} + a(b^{4j} + b^{-2} + b^{4j-3} + b^{4j+1})\right]b^4\\ &= (b\alpha)^2b^4,\end{aligned}$$

since $a^2b^{4j} = 1$ or b^4. Thus $((b\alpha)^2, ab^{-3}) = b^4$. It now follows that $b^4 \in N$ and the theorem is established for $n = 2$.

Consider the case $n \geqslant 3$. Assume that N is a normal complement of G in G^*. Let $\lambda = 1 + b^{2^n} \in Z_2[G]$ and

$$K = \left\{\alpha_0 + \sum k_{ij}a^ib^j\lambda \in G^*: \alpha_0 = 1 \text{ or } b^{2^n}\right\}.$$

It is easy to see that $K \lhd G^*$ since λ is central in $Z_2[G]$. Moreover, K has exponent 2 since $\lambda^2 = 0$. Consequently, KN/N is normal and has exponent 2 in $G^*/N \simeq G$. Each of our groups G has only one normal subgroup with exponent 2, namely $\langle b^{2^n}\rangle$, and thus $K \subseteq \langle b^{2^n}\rangle N$.

Let $M = \langle b^8\rangle N$. Clearly $M \lhd G^*$ and $K \subseteq M$ since $n \geqslant 3$. Since $N \cap G = \langle 1\rangle$, $\{a^ib^j: 0 \leqslant i \leqslant 1, 0 \leqslant j \leqslant 7\}$ is a complete set of coset representatives for M in G^*. It follows that G^*/M is the dihedral group of order 16. Thus for any $\alpha M \in G^*/M$, either α^2M or $(b\alpha)^2M$ is central in G^*/M. Thus for any $\alpha \in G^*$, either $(\alpha^2, \beta) \in M$ for all $\beta \in G^*$, or $((b\alpha)^2, \beta) \in M$ for all $\beta \in G^*$.

Let $\alpha = 1 + a + b \in G^*$. It is easy to compute that

$$\begin{aligned}\alpha^2 &= 1 + a^2 + b^2 + a(b + b^{j2^n-1})\\ &= \left[b^2 + a(b + b^{j2^n-1})\right]\left[1 + (b^2 + a(b + b^{j2^n-1}))^{-1}(1 + a^2)\right]\\ &= \left[b^2 + a(b + b^{j2^n-1})\right]k,\end{aligned}$$

where $k \in K$ since $1 + a^2 = 0$ or λ.

Then

$$\begin{aligned}(\alpha^2)^{ab^{-2}} &= \left[b^{-2} + a(b^{j2^n-1} + b)b^{-4}\right]k^{ab^{-2}}\\ &= \left[b^2 + a(b^{j2^n-1} + b)\right]kk^{-1}b^{-4}k^{ab^{-2}} = \alpha^2b^{-4}m,\end{aligned}$$

where $m \in M$ since $k \in K \subseteq M \lhd G^*$. Thus $(\alpha^2, ab^{-2}) \in b^{-4}M$. Also it is easy to compute that

$$\begin{aligned}(b\alpha)^2 &= b^2 + b^4 + a^2b^{j2^n} + a(b^{-2} + b^{j2^n} + b^{j2^n+1} + b^{j2^n-3})\\ &= \gamma\left[1 + \gamma^{-1}(1 + a^2b^{j2^n})\right],\end{aligned}$$

where

$$\gamma = 1 + b^2 + b^4 + a(b^{-2} + b^{j2^n} + b^{j2^n+1} + b^{j2^n-3}).$$

Thus $(b\alpha)^2 = \gamma k$, $k \in K$. Then

$$\begin{aligned}\left[(b\alpha)^2\right]^{ab^{-3}} &= \left[1 + b^{-2} + b^{-4} + a(b^2 + b^{j2^n} + b^{-1} + b^3)b^{j2^n-6}\right]k^{ab^{-3}}\\ &= \left[b^4 + b^2 + 1 + a(b^{j2^n} + b^{-2} + b^{j2^n-3} + b^{j2^n+1})\right]b^{-4}k^{ab^{-3}}\\ &= (b\alpha)^2 b^{-4}m, \quad \text{where } m \in M.\end{aligned}$$

Thus $((b\alpha)^2, ab^{-3}) \in b^{-4}M$. It now follows that $b^{-4} \in M$, a contradiction, and the proof is complete.

It is known that ten of the fourteen groups of order 16 belong to L_2 (see Johnson [1], and L. E. Moran and R. N. Tench [2]). The above theorem proves that three of the other groups are not in L_2. The following theorem proves that the remaining group of order 16 is in L_2.

THEOREM 2. *The group* $G = \langle a, b\colon a^2 = 1, b^8 = 1, a^{-1}ba = b^5\rangle$ *belongs to* L_2.

PROOF. Each $\alpha \in G^*$ can be written uniquely as

$$\alpha = u_0 + u_1 b + u_2 b^2 + u_3 b^3 + u_4 a + u_5 ab + u_6 ab^2 + u_7 ab^3,$$

where each $u_i = 0, 1, b^4$ or $1 + b^4$.

Let $-\colon Z_2[G] \to Z_2$ be the augmentation homomorphism, i.e., $\bar{\alpha}$ is the coefficient-sum of α. Let

$$H = \left\{\alpha \in G^*\colon \bar{u}_1 + \bar{u}_3 + \bar{u}_5 + \bar{u}_7 = 0, \bar{u}_4 + \bar{u}_5 + \bar{u}_6 + \bar{u}_7 = 0, \bar{u}_2 + \bar{u}_6 = 0\right\}.$$

Of course $\alpha \in G^*$ implies $\bar{u}_0 + \bar{u}_1 + \cdots + \bar{u}_7 = 1$, and so $\alpha \in H$ implies $\bar{u}_0 + \bar{u}_4 = 1$. A tedious but straightforward calculation shows that $\alpha, \beta \in H$ implies $\alpha\beta \in H$, and thus H is a subgroup of G^*. Clearly $o(H) = 2^{12}$ and a complete set of coset representatives for H in G^* is $\{1, b, \ldots, ab^3\}$. Let $\lambda = 1 + b^4$ and let

$$\begin{aligned}M = \Big\{\alpha \in G^*\colon \alpha = 1 &+ (c_1 + c_3 + c_5 + c_7)\lambda + c_1 b\lambda + c_2 b^2\lambda\\ &+ c_3 b^3\lambda + c_4 a\lambda + c_5 ab\lambda + c_6 ab^2\lambda + c_7 ab^3\lambda, \text{ each } c_i = 0 \text{ or } 1\Big\}.\end{aligned}$$

Clearly λ is central in $Z_2[G]$ and $\lambda^2 = 0$, so M is a subgroup of H. Another calculation shows that for $h \in H$,

$$\begin{aligned}h^2 = 1 &+ (u_4u_5 + u_6u_7)b\lambda + (\bar{u}_3 + \bar{u}_5)b^2\lambda + (u_4u_7 + u_5u_6)b^3\lambda + (u_1u_7 + u_3u_5)a\lambda\\ &+ (u_1u_4 + u_3u_6)ab\lambda + (u_1u_5 + u_3u_7)ab^2\lambda + (u_1u_6 + u_3u_4)ab^3\lambda \in M.\end{aligned}$$

Note that $u_iu_j = 0, 1, b^4$ or $1 + b^4$; $u_iu_j\lambda = 0$ or λ; and

$$(u_iu_j + u_ku_l)\lambda = (\bar{u}_i\bar{u}_j + \bar{u}_k\bar{u}_l)\lambda.$$

Since H is a 2-group, the Frattini subgroup of H, $\phi(H)$, is generated by H^2 and thus $\phi(H) \subseteq M$.

Clearly $b^4 \in H$ but $b^4 \notin M$. We can build a basis for H by first finding a basis for M over $\phi(H)$, adjoining b^4, and then completing a basis for H. Thus $H = \langle b^4, h_1, \ldots, h_s, \phi(H)\rangle$. Let $N = \langle h_1, \ldots, h_s, \phi(H)\rangle$. We will show that N is a normal complement of G in G^*. Clearly $[H\colon N] = 2$ and thus $o(N) = 2^{11}$, as required. Since $G \cap H = \langle 1, b^4\rangle$, we have $G \cap N = \langle 1\rangle$. Thus $o(GN) = 2^{15}$ and $GN = G^*$. All that remains is to show that $N \lhd G^*$. Since G is a complete set of coset representatives for N in G^*, it suffices to show that $(N, a), (N, b) \subset N$. In

fact, we will show that $h \in H$ implies (h, a), $(h, b) \in M$. A straightforward calculation shows that $h^a = h + (u_1 b + u_3 b^3 + u_5 ab + u_7 ab^3)\lambda$, and thus $(h, a) = 1 + h^{-1}(u_1 b + u_3 b^3 + u_5 ab + u_7 ab^3)\lambda$.

Since h^{-1} is an odd power of h and $h^2\lambda = \lambda$, we have

$$\begin{aligned}(h, a) &= 1 + h(u_1 b + u_3 b^3 + u_5 ab + u_7 ab^3)\lambda \\ &= 1 + (u_0u_1 + u_2u_3 + u_4u_5 + u_6u_7)b\lambda + (u_0u_3 + u_1u_2 + u_4u_7 + u_5u_6)b^3\lambda \\ &\quad + (u_0u_5 + u_2u_7 + u_1u_4 + u_3u_6)ab\lambda \\ &\quad + (u_0u_7 + u_2u_5 + u_3u_4 + u_1u_6)ab^3\lambda \in M.\end{aligned}$$

A similar computation shows that $(h, b) \in M$, and the proof is complete.

The above proof shows that a normal complement N exists, but gives little information about how to find N. In fact, an explicit computer calculation shows that one satisfactory N is the normal subgroup of G^* generated by $b^2 + a + ab^2$, $b + a + ab$, $1 + ab + ab^3$ and $b + a + ab^3$.

References

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2. L. E. Moran and R. N. Tench, *Normal complements in* mod *p-envelopes*, Israel J. Math. **27** (1977), 331–338.

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