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THE LASKERIAN PROPERTY, POWER SERIES RINGS AND NOETHERIAN SPECTRA

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ABSTRACT. We show that if the power series ring $R[[X]]$ in one indeterminate over a commutative ring R with identity is Laskerian, then R is Noetherian. On the other hand, if $R[[X]]$ is a ZD-ring, then R has Noetherian spectrum, but R need not be Noetherian. We show that, in general, a Laskerian ring has Noetherian spectrum.

Let R be a commutative ring with identity. An ideal Q of R is *primary* if each zero divisor of the ring R/Q is nilpotent, and Q is *strongly primary* if Q is primary and contains a power of its radical. In the terminology of Bourbaki [**B**, Ch. IV, pp. 295, 298], the ring R is *Laskerian* if each ideal of R is a finite intersection of primary ideals, and R is *strongly Laskerian* if each ideal of R is a finite intersection of strongly primary ideals. It is well known that

$$\text{Noetherian} \Rightarrow \text{strongly Laskerian} \Rightarrow \text{Laskerian},$$

and Evans in [**E**] showed that a Laskerian ring is what he calls a ZD-*ring* (for *zero-divisor ring*), which is defined as follows. A ring R is a ZD-*ring* if the set of zero divisors on the R-module R/A is a finite union of prime ideals for each ideal A of R. In [**HO**], Heinzer and Ohm proved that R is Noetherian if $R[X]$ is a ZD-ring, and hence the conditions Noetherian, strongly Laskerian, Laskerian and ZD are equivalent in $R[X]$. We investigate here relationships among these four conditions in the power series ring $R[[X]]$. We prove in Theorem 1 that $R[[X]]$ Laskerian implies R is Noetherian. We then give an example to show that R need not be Noetherian if $R[[X]]$ is a ZD-ring. On the other hand, R has Noetherian spectrum if $R[[X]]$ is a ZD-ring (Theorem 2). The paper concludes with the result that a Laskerian ring has Noetherian spectrum.

THEOREM 1. *Let R be a commutative ring with identity. The power series ring $R[[X]]$ in one variable over R is Laskerian if and only if R is Noetherian.*

PROOF. It is enough to prove the "only if" part of the theorem. Assume, to the contrary, that $R[[X]]$ is Laskerian and R is not Noetherian. By [**AGH**, Theorem 2.3], there exists a prime ideal P of R such that $PR[[X]]$, the extension of P to $R[[X]]$, is properly contained in $P[[X]]$, the set of power series all of whose

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coefficients belong to P. Pick $f \in P[[X]]$, $f \notin PR[[X]]$, and let $A = PR[[X]] + (Xf)$. Let $A = \bigcap_{i=1}^{n} Q_i$ be a shortest primary representation of A, where Q_i is P_i-primary. Thus $\bigcup_{i=1}^{n} P_i$ is the set of zero divisors on R/A. We note that Nakayama's Lemma applied to the ring $R[[X]]/PR[[X]]$ shows that $f \notin A$, and hence X is a zero divisor on R/A. Choose j so that $X \in P_j$, and let k be such that $X^k \in Q_j$. Then $Q_j \supseteq (PR[[X]], X^k) \supset P[[X]]$. Since $P[[X]]$ is prime and contains A, it follows that $Q_t \subseteq P[[X]]$ for some t, and hence $Q_t \subset Q_j$. This contradiction to the irredundance of the representation $\bigcap_{i=1}^{n} Q_i$ completes the proof of Theorem 1.

In contrast to the polynomial ring case, we proceed to give an example showing that $R[[X]]$, a ZD-ring, does not imply that R is Noetherian. Thus, assume that F and K are fields of nonzero characteristic p, that F is a subfield of K and that K/F is infinite dimensional, purely inseparable, and of finite exponent e. Let V be a rank-one discrete valuation ring of the form $K + M$, where M is the maximal ideal of V, and let $R = F + M$. Clearly V is the integral closure of R, and since K/F is infinite dimensional, the domain R is not Noetherian. Now $V[[X]]$ is a two-dimensional regular local ring, and $V[[X]]$ and $R[[X]]$ have the same quotient field since the conductor $M[[X]]$ of $R[[X]]$ in $V[[X]]$ is nonzero. Moreover, $f^{p^e} \in R[[X]]$ for each $f \in V[[X]]$ so that $V[[X]]$ is the integral closure of $R[[X]]$, and these two rings have homeomorphic spectra. In particular, $R[[X]]$ is a two-dimensional quasi-local domain with Noetherian spectrum, and hence is a ZD-ring.

The next result shows that one aspect of the preceding example carries over to the general case.

THEOREM 2. *If $R[[X]]$ is a ZD-ring, then R has Noetherian spectrum.*

PROOF. We establish the contrapositive. Thus, assume that there exists an infinite strictly ascending sequence $A_1 \subset A_2 \subset \cdots$ of radical ideals of R. For each i, let P_i be a prime ideal of R such that $A_i \subseteq P_i$ and $A_{i+1} \not\subseteq P_i$. Let $\overline{A}_{i+1}$ denote the canonical image of A_{i+1} in R/P_i. If $\overline{A}_{i+1}$ is all of R/P_i, then let $Q_i = P_i[[X]]$ in $R[[X]]$. If $\overline{A}_{i+1}$ is a proper ideal in R/P_i, let $a_i \in A_{i+1} \setminus P_i$ and consider the ideal generated by $\bar{a}_i + X$ in $(R/P_i)[[X]]$. Since R/P_i is an integral domain, the ideal $(\overline{a}_i + X)$ does not meet the multiplicative system $\{\overline{a}_i^n\}_{n=1}^{\infty}$ in R/P_i. Hence there exists a prime ideal $\overline{Q}_i$ of $(R/P_i)[[X]]$ that contains $\overline{a}_i + X$ and does not meet $\{\overline{a}_i^n\}$. Let Q_i be the inverse image of $\overline{Q}_i$ in $R[[X]]$. We observe that Q_i is a prime ideal such that $P_i[[X]] \subseteq Q_i$, $a_i + X \in Q_i$ and $a_i \notin Q_i$ so that $A_{i+1} \not\subseteq Q_i$ and $X \notin Q_i$.

Consider the set $\{Q_i\}$ of prime ideals defined above. We show first that there is no containment relation between Q_i and Q_j for $i < j$. Note that $Q_j \not\subseteq Q_i$ since $A_j \subseteq Q_j$ and $A_j \not\subseteq Q_i$. To show that $Q_i \not\subseteq Q_j$, we consider separately the cases $\overline{A}_{i+1} = R/P_i$ and $\overline{A}_{i+1} \neq R/P_i$. If $\overline{A}_{i+1} = R/P_i$, then $Q_i = P_i[[X]]$ and $Q_j + Q_i = R[[X]]$ since $Q_i + Q_j \supseteq P_i[[X]] + A_{i+1}R[[X]] = R[[X]]$. Hence $Q_i \not\subseteq Q_j$ in this case. If, however, $\overline{A}_{i+1} \neq R/P_i$, then $a_i + X \in Q_i$ and $a_i + X \notin Q_j$ since $a_i \in A_{i+1} \subseteq Q_j$ and $X \notin Q_j$ (note that the relations $A_{i+1} \subseteq Q_j$ and $X \notin Q_j$ are true no matter whether $\overline{A}_{j+1} = R/P_j$ or $\overline{A}_{j+1} \neq R/P_j$). Let $B = \bigcap_{i=1}^{\infty} Q_i$. We observe that this intersection is irredundant. To prove that Q_n is irredundant, we note that

$Q_1 \cdots Q_{n-1}A_{n+1}$ is contained in $\bigcap_{i \neq n} Q_i$, but is not contained in Q_n. Irredundancy of the representation $\bigcap_1^\infty Q_i$ easily implies that $\bigcup_1^\infty Q_i$ is the set of zero divisors on R/B. To complete the argument, we prove that $\bigcup_1^\infty Q_i$ is not a finite union of prime ideals, and hence $R[[X]]$ is not a ZD-ring. Suppose not, and let $\bigcup_1^\infty Q_i = M_1 \cup \cdots \cup M_k$. Each Q_i is contained in some M_t, and since the set $\{Q_i\}$ is infinite, some M_t contains infinitely many of the primes Q_i–say M_t contains Q_i and Q_j, where $i < j$. If $Q_i = P_i[[X]]$, we obtain the contradiction that $R[[X]] = Q_i + Q_j \subseteq M_t$, and if $Q_i \neq P_i[[X]]$, then $X = (a_i + X) + (-a_i) \in Q_i + Q_j \subseteq M_t$, contrary to the fact that $X \notin \bigcup_{i=1}^\infty Q_i$.

We conclude the paper with a proof that a Laskerian ring has Noetherian spectrum. If R is Laskerian, it is clear that each ideal of R has only finitely many minimal prime divisors. Since a ring with the latter property has Noetherian spectrum if and only if the ascending chain condition for prime ideals (a.c.c.p.) is satisfied in R (**[M**, Sätze 15, 16] or **[OP]**), to prove that R has Noetherian spectrum, it suffices to prove that a.c.c.p. is satisfied in R. The next result is well known for a Noetherian ring **[ZS**, Theorem 20, p. 229]; we extend to the case of a Laskerian ring.

PROPOSITION 3. *Let P be a prime ideal of R, a Laskerian ring, and let* $(0) = \bigcap_{i=1}^n Q_i$ *be a shortest primary representation of* (0) *in R. The intersection of the set of P-primary ideals of R is the intersection of the family of components Q_i that are contained in P.*

PROOF. By standard techniques of localization, it is enough to prove that if R is quasi-local with maximal ideal P, then the intersection of the set of P-primary ideals is (0). Thus, take $x \in P$, $x \neq 0$. We have $x \notin xP$, and hence P is the set of zero divisors on R/xP. Therefore P is a belonging prime of xP, and there exists a P-primary ideal Q that does not contain x. As x is arbitrary it follows that the intersection of the set of P-primary ideals is (0).

THEOREM 4. *A Laskerian ring has Noetherian spectrum.*

PROOF. Let R be a Laskerian ring and assume that R does not have Noetherian spectrum. Then there exists an infinite strictly ascending sequence

$$P_0 \subset P_1 \subset P_1' \subset P_2 \subset P_2' \subset \cdots$$

of proper prime ideals of R. By passage to R/P_0, we assume without loss of generality that R is an integral domain. We prove by induction that there exist ideals $Q_1, \ldots, Q_n, A_n, B_n$ of R and elements $x_1, x_2, \ldots, x_n$ of R with the following properties.

(1) Q_i is P_i-primary for each i, and $A_n = Q_1 \cap \cdots \cap Q_n$.

(2) For $1 \leqslant i \leqslant n$, $x_i \in \bigcap_{j \neq i} Q_j$ and $x_i \notin Q_i$.

(3) $(x_1, \ldots, x_n) \subseteq B_n$, $A_n \nsubseteq B_n$, and each belonging prime of B_n is contained in P_n'.

For $n = 1$, we take $Q_1 = P_1$. Proposition 3 implies that there exists a P_1'-primary ideal Q_1' that does not contain P_1. Pick $x_1 \in Q_1'$, $x_1 \notin Q_1$, and define B_1 to be $x_1R_{P_1'} \cap R$. We have $B_1 \nsupseteq A_1 = Q_1$ since $B_1R_{P_1'} = x_1R_{P_1'} \subseteq Q_1'R_{P_1'}$ and $P_1 \nsubseteq Q_1'R_{P_1'}$, and the other conditions of (1)–(3) are clearly satisfied.

Assume that $Q_1, \ldots, Q_n, A_n, B_n, x_1, \ldots, x_n$ are given satisfying (1)–(3). Choose $y_{n+1} \in A_n, y_{n+1} \notin B_n$. Applying Proposition 3 to the Laskerian ring $R_{P_{n+1}}/B_nR_{P_{n+1}}$, we conclude that there exists a P_{n+1}-primary ideal Q_{n+1} containing B_n such that $y_{n+1} \notin Q_{n+1}$. We define $A_{n+1} = A_n \cap Q_{n+1}$. Note that $A_{n+1} \not\subseteq B_n$, for $A_n \not\subseteq B_n$ and Q_{n+1} is not contained in the set of zero divisors on R/B_n. To define x_{n+1} and B_{n+1}, first observe that Proposition 3 implies that $B_n R_{P'_{n+1}} = \bigcap_{\lambda \in \Lambda}(B_n, y_{n+1}C_\lambda) R_{P'_{n+1}}$, where $\{C_\lambda\}_{\lambda \in \Lambda}$ is the set of P'_{n+1}-primary ideals. Since $A_{n+1} \not\subseteq B_n$ and since $B_n = B_n R_{P'_{n+1}} \cap R$, there exists $\lambda \in \Lambda$ such that $A_{n+1} \not\subseteq (B_n, y_{n+1}C_\lambda) R_{P'_{n+1}} \cap R$. Choose $r \in C_\lambda$, $r \notin P_{n+1}$, and define $x_{n+1} = y_{n+1}r$, $B_{n+1} = (B_n, x_{n+1}) R_{P'_{n+1}} \cap R$. Since $y_{n+1} \notin Q_{n+1}$ and $r \notin P_{n+1}$, we have $x_{n+1} \notin Q_{n+1}$. Thus, (1) and (2) are satisfied for $Q_1, \ldots, Q_{n+1}$ and $x_1, x_2, \ldots, x_{n+1}$. Moreover, (3) is satisfied for B_{n+1} by choice of x_{n+1} and B_{n+1}. By induction, we conclude that there exist infinite sequences $\{Q_i\}_1^\infty$, $\{A_i\}_1^\infty$, $\{x_i\}_1^\infty$ and $\{B_i\}_1^\infty$ so that conditions (1), (2) and (3) are satisfied for all n.

We define $A = \bigcap_{i=1}^\infty Q_i$, and we note that this representation is irredundant since $x_n \in \bigcap_{j \neq n} Q_j$ and $x_n \notin Q_n$ for each n. Moreover, $A\colon (x_n) = Q_n\colon (x_n)$ is P_n-primary for each n. This implies that A admits no representation as a finite intersection of primary ideals, for if it did–say $A = \bigcap_{i=1}^k H_i$ is a shortest representation, where H_i is M_i-primary–then a standard argument shows that $\{M_i\}_{i=1}^k$ is the set of prime ideals of R realizable as the radical of an ideal of the form $A\colon (x)$. Therefore R is not Laskerian, and this completes the proof of Theorem 4.

We remark that the ZD-property implies neither (1) ascending chain condition for prime ideals (a.c.c.p.), nor (2) that each ideal has only finitely many minimal primes. For example, each valuation ring is a ZD-ring, and a valuation ring need not satisfy a.c.c.p. For (2), let $\{X_i\}_{i=1}^\infty$ be a set of indetminates over the field K, let $D = \bigcup_{n=1}^\infty K[[X_1, \ldots, X_n]]$ and let $R = D/(\{X_iX_j | i \neq j\})$. The ring R is one-dimensional quasi-local with maximal ideal $M = (\{x_n\}_1^\infty)$, and $\{P_i\}_{i=1}^\infty$ is the set of minimal primes of R, where $P_i = (\{x_j | j \neq i\})$. The union of each infinite subset of $\{P_i\}_{i=1}^\infty$ is M, so R is a ZD-ring, but (0) has infinitely many minimal primes.

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