PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, May 1980

ON J-SELFADJOINT EXTENSIONS OF J-SYMMETRIC OPERATORS

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ABSTRACT. A short proof is given (via the theory of conjugate-linear operators) of the fact that every J-symmetric operator in a Hilbert space $\mathcal{H}$ has a J-selfadjoint extension in $\mathcal{H}$.

An operator J on a complex Hilbert space $\mathcal{H}$ is called a conjugation if it is an involution and

$$(Jx, Jy) = (y, x) \tag{1}$$

for all x and y in $\mathcal{H}$. Following I. M. Glazman [3], a linear operator A in $\mathcal{H}$ with domain $\mathfrak{D}(A)$ dense in $\mathcal{H}$ is called J-symmetric if

$$(Ax, Jy) = (x, JAy) \tag{2}$$

for all x and y in $\mathfrak{D}(A)$. Clearly, A is J-symmetric if and only if $JAJ \subset A^*$. If $JAJ = A^*$, then A is said to be J-selfadjoint.

Now, it is well known that symmetric linear operators in $\mathcal{H}$ need not possess selfadjoint extensions; indeed, it is a basic fact in the theory that these operators are precisely those with unequal deficiency indices (see [1, Corollary 13, p. 1230]). It is thus rather surprising to learn that the following result is known for J-symmetric operators.

THEOREM A [2]. *Every J-symmetric operator in $\mathcal{H}$ has a J-selfadjoint extension in $\mathcal{H}$.*

The method of proof used in [2] makes use of certain properties of the graphs of the operators concerned. We return to this method later. Our main objective here is to give a simplified proof of this result.

The method that we use is based upon a consideration of the analogous theory for conjugate-linear operators in $\mathcal{H}$. Here, an operator T, with domain $\mathfrak{D}(T)$ dense in $\mathcal{H}$, is said to be conjugate-linear if

$$T(\alpha x + \beta y) = \bar{\alpha} Tx + \bar{\beta} Ty$$

for all scalars α and β, and all x and y in $\mathfrak{D}(T)$. By analogy with the corresponding theory for linear operators, we define $\mathfrak{D}(T^*)$ to be the set of elements y in $\mathcal{H}$ for which there corresponds an element z in $\mathcal{H}$ (necessarily unique) such that $(Tx, y) = (z, x)$ holds for all x in $\mathfrak{D}(T)$. The adjoint, T^*, of T is defined on $\mathfrak{D}(T^*)$ by the equation $T^*y = z$. Thus

Received by the editors April 18, 1979.
AMS (MOS) subject classifications (1970). Primary 47B25.
Key words and phrases. J-symmetric operators, J-selfadjoint extension, conjugate-linear operator.

$$(Tx, y) = (T^*y, x) \tag{3}$$

for all $x \in \mathfrak{D}(T)$ and $y \in \mathfrak{D}(T^*)$. The conjugate-linear operator T is said to be symmetric if

$$(Tx, y) = (Ty, x) \tag{4}$$

for all x and y in $\mathfrak{D}(T)$; thus T is symmetric if and only if $T \subset T^*$. If $T = T^*$, then T is called selfadjoint. Finally, we say that T is maximal symmetric if T is symmetric and possesses no proper symmetric extensions. The usual maximality argument, using Zorn's lemma, shows that every symmetric conjugate-linear operator has a maximal symmetric extension. Thus, if we observe that the linear operator A is J-symmetric (J-selfadjoint) if and only if the conjugate-linear operator JA is symmetric (selfadjoint), then Theorem A may be equivalently formulated as,

THEOREM B. *Every maximal symmetric conjugate-linear operator is selfadjoint.*

PROOF. If we assume to the contrary, that the theorem is not true, then there is a conjugate-linear operator T that is maximal symmetric, but not selfadjoint.

Let $y \in \mathfrak{D}(T^*) \setminus \mathfrak{D}(T)$, and define an extenson T_1 of T by

$$\mathfrak{D}(T_1) = \mathfrak{D}(T) + \{\alpha y\}, \qquad \alpha \in \mathbf{C},$$

and, for $x = p + q \in \mathfrak{D}(T_1)$ where $p \in \mathfrak{D}(T)$ and $q \in \{\alpha y\}$ set

$$T_1 x = Tp + T^*q.$$

Then, for $x = p_1 + q_1$ and $y = p_2 + q_2$ in $\mathfrak{D}(T_1)$, where $q_1 = \alpha_1 y$ and $q_2 = \alpha_2 y$ for some complex scalars α_1 and α_2, we have

$$\begin{aligned}(T_1x, y) &= (Tp_1, p_2) + (Tp_1, q_2) + (T^*q_1, p_2) + (T^*q_1, q_2)\\ &= (Tp_2, p_1) + (T^*q_2, p_1) + (Tp_2, q_1) + \bar{\alpha}_1\bar{\alpha}_2(T^*y, y)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (3) and (4)}\\ &= (T_1 y, x).\end{aligned}$$

Thus T_1 is a proper symmetric extension of T, in contradiction to the maximality of T. The proof is complete. □

As a final result we have:

THEOREM C. *Let* $\mathfrak{H}_2 = \mathfrak{H} \oplus \mathfrak{H}$, *and let* $G_A = \{[x, y] \in \mathfrak{H}_2 : y = Ax\}$ *denote the graph of the closed J-symmetric operator A. Let* T: $\mathfrak{H}_2 \to \mathfrak{H}_2$ *be defined by* $T[x, y] = [Jy, -Jx]$, *and set* $D = \mathfrak{H}_2 \ominus [G_A \oplus TG_A]$. *Then*

(i) *there exists a subspace* X *in* $\mathfrak{H}_2$ *such that* $X \oplus TX = D$, *and the set* $G_A \oplus X$ *is the graph of a J-selfadjoint extension of A*,

(ii) *every J-selfadjoint extension of A has the form given in* (i).

PROOF. Part (i) is precisely the result proved in **[2]**. Alternatively, part (i) may be derived from Theorem A, and part (ii), which we now prove. Let B be an arbitrary J-selfadjoint extension of A, with graph G_B, and set $X = G_B \ominus G_A$. Since $G_{JB^*J} = TG_B^{\perp}$, and B is J-selfadjoint, it is clear that $G_B \oplus TG_B = \mathfrak{H}_2$, and thus that

$$X \oplus G_A \oplus T(X \oplus G_A) = \mathfrak{H}_2. \tag{5}$$

Finally, as $X \perp G_A$, it is not hard to see that $TX \perp TG_A$. The result now follows from (5). $\square$

References

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