AN EXAMPLE OF A LIMINAL C*-ALGEBRA

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ABSTRACT. For each countable ordinal $\gamma$ there exists a unital separable liminal C*-algebra $A_\gamma$ with the property that if $(I_\rho_{\rho=1})$ is any composition sequence of $A_\gamma$ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff, then $\beta > \gamma + 1$. Moreover, there is a composition sequence $(I_\rho_{\rho=1})$ of $A_\gamma$ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff.

1. Introduction. In [2, 4.7.25] J. Dixmier posed the following problem: Construct an example of a liminal C*-algebra which does not admit any finite composition sequence $(I_\rho)$ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff. Blackadar solved this problem in [1, p. 325]. The purpose of this note is to show that there are examples with the above property with respect to any countable ordinal. Specifically, we prove the following theorem.

THEOREM. For each countable ordinal $\gamma$ there exists a unital separable liminal C*-algebra $A_\gamma$ with the property that if $(I_\rho_{\rho=1})$ is any composition sequence of $A_\gamma$ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff, then $\beta > \gamma + 1$. Moreover, there is a composition sequence $(I_\rho_{\rho=1})$ of $A_\gamma$ such that the spectra of the quotients $I_{\rho+1}/I_\rho$ are Hausdorff.

2. The two-point $T_1$ compactification. In this section we present some topological results that are necessary for the proof of our theorem.

Let $X$ be a topological space that is at least $T_1$ and let $\infty_1$, $\infty_2$ be points not in $X$. Let $X_1$ be the set $X \cup \{\infty_1, \infty_2\}$ together with the topology whose members are all subsets of $U$ of $X$, such that (i) if $U \cap \{\infty_1, \infty_2\} = \emptyset$, then $U$ is an open subset of $X$, (ii) if $U \cap \{\infty_1, \infty_2\} \neq \emptyset$, then the complement of $U \cup \{\infty_1, \infty_2\}$ is a closed compact subset of $X$. Clearly, $X_1$ is a compact $T_1$ space and if $X$ is not compact, then $X_1$ is not Hausdorff. We call $X_1$ the two-point $T_1$ compactification of $X$.

Next suppose $\beta$ is an ordinal. A topological space $X$ is said to have a Hausdorff decomposition of length $\beta$ if there exists a generalized sequence $\{U_\rho\}_{\rho=1}^\beta$ of open subsets of $X$ with the following properties. (i) $U_\rho \subseteq U_{\rho+1}$, $\rho < \beta$, (ii) $U_1$ and $U_{\rho+1} \setminus U_\rho$, $\rho < \gamma$, are Hausdorff, (iii) if $\rho < \beta$ is a limit ordinal, then $U_\rho = \cup_{\delta < \rho} U_\delta$, (iv) $U_\beta = X$.

Now assume $X$ is a denumerable discrete set and let $X_1$ denote the two-point $T_1$ compactification of $X$. Next let $\beta$ be a countable ordinal. Suppose that for all $\rho < \beta$, $X_\rho$ has already been defined. If $\beta$ is not a limit ordinal, let $X_\beta$ be the two-point $T_1$ compactification of the disjoint union $\cup_{n=1}^\infty Y_\rho$, where each $Y_\rho$ is a

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copy of \( X_{\beta-1} \). If \( \beta \) is a limit ordinal, let \( X_{\beta} \) be the two-point \( T_1 \) compactification of the disjoint union \( \bigcup_{\rho<\beta} X_{\rho} \). We have defined by induction a compact \( T_1 \) topological space \( X_{\beta} \) for each countable ordinal \( \beta \).

2.1. Lemma. Let \( \beta \) be a countable ordinal and let \( X_{\beta} \) be the topological space defined above. Then \( X_{\beta} \) does not admit any Hausdorff decomposition of length less than or equal to \( \beta \) but does admit one of length \( \beta + 1 \).

Proof. Clearly the assertion holds for the topological space \( X_{\beta} \). Now suppose the assertion holds for all topological spaces \( X_{\rho}, \rho < \beta \). First, assume \( \beta \) is not a limit ordinal. Recall that \( X_{\beta} \) is the two-point \( T_1 \) compactification of the disjoint union \( \bigcup_{n=1}^{\infty} Y_n \), where each \( Y_n \) is a copy of \( X_{\beta-1} \). Let \( \{V_{\rho}\}_{\rho=1}^{\beta} \) be a Hausdorff decomposition of \( X_{\beta-1} \) of length \( \beta \). Set \( U_{\beta+1} = X_{\beta} \) and \( U_{\rho} = \bigcup_{n=1}^{\infty} V_{\rho,n}, \rho < \beta \), where \( V_{\rho,n} \) is the copy of \( V_{\rho} \) in \( Y_n \). Clearly, \( \{U_{\rho}\}_{\rho=1}^{\beta+1} \) is a Hausdorff decomposition of \( X_{\beta} \) of length \( \beta + 1 \). Now suppose \( \{U_{\rho}\}_{\rho=1}^{\gamma} \) is a Hausdorff decomposition of \( X_{\beta} \) of length \( \gamma \). Assume \( \gamma < \beta \). If \( \gamma < \beta \), then \( \{U_{\rho} \cap Y_n\}_{\rho=1}^{\gamma} \) is a Hausdorff decomposition of \( X_{\beta-1} \) of length less than or equal to \( \beta - 1 \), a contradiction. So assume \( \gamma = \beta \). If \( Y_n \cap U_{\beta-1} \neq \emptyset, n = 1, 2, 3, \ldots \), then there is a sequence in the complement of \( U_{\beta-1} \) that converges to two points in the complement of \( U_{\beta-1} \). This contradicts the fact that \( U_{\beta} \cap U_{\beta-1} \) is Hausdorff decomposition. Thus there is a \( Y_n \subset U_{\beta-1} \). But this implies that \( \{U_{\rho} \cap Y_n\}_{\rho=1}^{\beta} \) is a Hausdorff decomposition of \( Y_n \) of length \( \beta - 1 \), another contradiction. Thus \( \gamma > \beta + 1 \).

Suppose \( \beta \) is a limit ordinal. Recall that \( X_{\beta} \) is the two-point \( T_1 \) compactification of the disjoint union \( \bigcup_{\rho<\beta} X_{\rho} \). For each \( \rho < \beta \) let \( \{V_{\alpha,\rho}\}_{\alpha=1}^{\rho} \) be a Hausdorff decomposition of \( X_{\rho} \) of length \( \rho + 1 \). Set \( U_1 = \bigcup_{\rho=1}^{\beta} V_{\rho,1} \) and let \( \delta \) be an ordinal such that \( 1 < \delta < \beta \). Assume that \( U_{\alpha} \) has been defined for all \( \alpha < \delta \). If \( \delta \) is not a limit ordinal, then set

\[
U_\delta = \left( \bigcup_{\delta < \alpha < \beta} V_{\alpha,\delta} \right) \cup \left( \bigcup_{\alpha < \delta} X_{\alpha} \right).
\]

If \( \delta \) is a limit ordinal, then set \( U_\delta = \bigcup_{\alpha < \delta} U_{\alpha} \). Next set \( U_{\beta+1} = X_{\beta} \). Clearly \( \{U_{\rho}\}_{\rho=1}^{\beta+1} \) is a Hausdorff decomposition of \( X_{\beta} \) of length \( \beta + 1 \). Finally suppose \( \{U_{\rho}\}_{\rho=1}^{\gamma} \) is a Hausdorff decomposition of \( X_{\beta} \) of length \( \gamma \). Then \( \{U_{\rho} \cap X_{\alpha}\}_{\rho=1}^{\gamma} \) is a Hausdorff decomposition of \( X_{\alpha} \) of length greater than or equal to \( \alpha + 1 \) for all \( \alpha < \beta \). This implies \( \gamma > \alpha + 1 \) for all \( \alpha < \beta \). Thus \( \gamma > \beta \). If \( \beta = \gamma \), then \( X_{\beta} = U_{\beta} \) = \( \bigcup_{\rho<\beta} U_{\rho} \). Since \( X_{\beta} \) is compact, there exists a \( \rho < \beta \) such that \( X_{\beta} = U_{\rho} \). This means \( \{U_{\delta} \cap X_{\rho}\}_{\delta=1}^{\rho} \) is a Hausdorff decomposition of \( X_{\rho} \) of length \( \rho \), which contradicts our induction hypothesis. Thus our proof is complete.

3. The example. For each positive integer \( n \) let \( A_n \) denote a \( C^* \)-algebra with identity \( I_n \) and \( T_1 \) spectrum \( \tilde{A}_n \). Let \( p_n \) be a nontrivial projection in \( A_n \) such that \( \pi(p_n) \neq 0 \neq \pi(I_n - p_n) \) for all \( \pi \in \tilde{A}_n \); set \( q_n = I_n - p_n \). Let \( \Sigma \Theta A_n \) denote the direct sum of the sequence \( \{A_n\} \) and let \( \hat{A}((\{A_n, p_n\})) \) denote the set of all \( x \in \Sigma \Theta A_n \) with the following property. There exist complex numbers \( \lambda_1 \) and \( \lambda_2 \)
such that
\[ \lim_{n \to \infty} \| x(n) - \lambda_1 p_n - \lambda_2 q_n \| = 0. \]

3.1. Proposition. Let \( \mathfrak{A}(\{(A_n, p_n)\}) \) be defined as above. Then \( \mathfrak{A} = \mathfrak{A}(\{(A_n, p_n)\}) \) is a unital \( \text{C}^* \)-algebra whose spectrum \( \hat{\mathfrak{A}} \) is the two-point \( T_1 \) compactification of the disjoint union \( \bigcup_{n=1}^{\infty} \hat{A}_n \). Moreover if each \( A_n \) is separable and liminal, then \( \mathfrak{A} \) is separable and liminal.

Proof. It is straightforward to verify that \( \mathfrak{A} \) is a unital \( \text{C}^* \)-algebra and if each \( A_n \) is separable, it is clear that \( \mathfrak{A} \) is separable. Now let \( T \) be the two-point \( T_1 \) compactification of the disjoint union \( \bigcup_{n=1}^{\infty} \hat{A}_n \) and let \( \pi_1, \pi_2 \) be its points at infinity. Define the map \( \Psi : T \to \hat{\mathfrak{A}} \) by the following formula. For \( \pi_i \in T, i = 1, 2, \) and \( x \in \mathfrak{A} \) let
\[ \Psi(\pi_i)(x) = \lambda_i, \]
where \( \lambda_1 \) and \( \lambda_2 \) are the unique complex numbers for which
\[ \lim_{n \to \infty} \| x(n) - \lambda_1 p_n - \lambda_2 q_n \| = 0. \]

Clearly, \( \Psi(\pi_i), i = 1, 2, \) belong to \( \hat{\mathfrak{A}} \). If \( \pi \in A_n \) for some positive integer \( n \) and \( x \in \mathfrak{A} \), define \( \Psi(\pi)(x) = \pi(x(n)) \). Clearly \( \Psi \) is a one-to-one map of \( T \) into \( \hat{\mathfrak{A}} \). Now we wish to show that \( \Psi \) is an onto map. Let \( J_{\pi_i} \) be the set of all \( x \) in \( \mathfrak{A} \) for which there is a complex number \( \lambda \) such that \( \| x(n) - 0 \cdot p_n - \lambda q_n \| \to 0 \). Similarly define \( J_{\pi_i} \). Let \( p, q \in \mathfrak{A} \) be defined by \( p(n) = p_n, q(n) = q_n \). Note \( p \subseteq J_{\pi_2} \) and \( q \subseteq J_{\pi_1} \). Thus it easily follows that \( J_{\pi_i} \), \( i = 1, 2, \) are closed two-sided ideals of \( \hat{\mathfrak{A}} \) such that \( J_{\pi_1} + J_{\pi_2} = \mathfrak{A} \). Let \( \rho : \mathfrak{A} \to B(H) \) be an irreducible representation of \( \hat{\mathfrak{A}} \). Without loss of generality we may assume \( \rho|_{J_{\pi_1}} \) is nondegenerate. If \( \rho|_{J_{\pi_1}} = 0 \) for some positive integer \( n \) define \( \epsilon_n \) by the formula
\[ \epsilon_n = \left\{ \begin{array}{ll} I_{m}, & m < n, \\ q_m, & m > n. \end{array} \right. \]
Clearly, \( \{ \epsilon_n \} \) is an approximate identity for \( J_{\pi_1} \). Next let \( x \in J_{\pi_2} \) and \( \lambda \) be the unique complex number for which
\[ \| x(n) - 0 \cdot p_n - \lambda q_n \| \to 0. \]
By the assumption made on \( \rho \), \( \rho(x - \lambda \epsilon_n) \to 0 \). So for \( h \in H \), \( \rho(x)(h) = \lim_{n \to \infty} \rho(x - \lambda \epsilon_n)(h) = \lambda I_H(h) \) by virtue of \([2, 2.10.4]\). In particular, \( \rho(q) = I_H \). Let \( \lambda_1, \lambda_2 \) be the unique complex numbers for which
\[ \| x(n) - \lambda_1 p_n - \lambda_2 q_n \| \to 0. \]
It follows that \( \rho(x)(h) = \rho(q)(h) = \rho(xq)(h) = \lambda_2 h \), that is, \( \rho = \Psi(\pi_2) \). We have now proved that \( \Psi \) maps \( T \) onto \( \hat{\mathfrak{A}} \). This proves, by the way, that if each \( A_n \) is liminal, then \( \mathfrak{A} \) is liminal.

By virtue of \([2, 3.2.1]\) \( \Psi \) is a homeomorphism of \( \hat{A}_n \) onto \( \hat{\mathfrak{A}_n} \) and by \([2, 3.2.3]\) \( \hat{\mathfrak{A}_n} \) is an open and closed compact subset of \( \hat{\mathfrak{A}} \). Thus \( \Psi \) is a homeomorphism of the disjoint union \( \bigcup \hat{A}_n \) onto \( \bigcup \hat{\mathfrak{A}_n} \). It remains to be shown that the family of open neighborhoods of \( \pi_i \) is mapped onto the family of open neighborhoods of \( \rho_i = \Psi(\pi_i), i = 1, 2 \). Let \( U \) be an open subset of \( \hat{\mathfrak{A}} \) containing \( \rho_i \). Then there is a closed ideal \( J \) of \( A \) such that \( U = \hat{\mathfrak{A}}_J \). Thus there is an \( x \in J \) such that \( \rho_i(x) \neq 0 \). Let \( \lambda_1 \).
and \( \lambda_2 \) be the unique complex numbers for which
\[
\|x(n) - \lambda_1 p_n - \lambda_2 q_n\| \to 0.
\]
To say \( \rho_1(x) \neq 0 \) means \( \lambda_1 \neq 0 \). So \( xp \) has the property \( \rho_1(xp) = \lambda_1 \neq 0 \). Now suppose there is a subsequence \( \{n_k\} \) of positive integers and elements \( \pi_k \in \hat{A}_{n_k} \) for which \( \Psi(\pi_k) \notin U \). It follows that
\[
0 = \Psi(\pi_k)(xp(n_k)) = \pi_k(xp(n_k) - \lambda_1 p(n_k) + \lambda_1 \pi_k(p(n_k))).
\]
Since \( \pi_k(p(n_k)) \neq 0 \), \( \|\pi_k(p(n_k))\| = 1 \). Thus
\[
|\lambda_1| = |\pi_k(x(n_k) - \lambda_1 p_{n_k} - \lambda_2 q_{n_k})p_{n_k}| \to 0,
\]
a contradiction. So there exists a positive integer \( N \) such that \( \Psi(\hat{A}_n) \subset U \) for all \( n > N \). Now it is straightforward to verify that
\[
\Psi^{-1}(U) \cup \{\pi_1, \pi_2\} = \left( \bigcup_{n=1}^{N} \Psi^{-1}(\hat{A}_n) \cap U \right) \cup \left( \bigcup_{n=N+1}^{\infty} \hat{A}_n \right) \cup \{\pi_1, \pi_2\}
\]
so clearly \( \Psi^{-1}(U) \) is an open neighborhood of \( \pi_1 \). Similarly, \( \Psi^{-1}(U) \) is an open neighborhood of \( \pi_2 \) whenever \( U \) is an open neighborhood of \( \rho_2 \).

Finally, let \( V \) be an open subset of \( T \) containing \( \pi_1 \). There is a positive integer \( N \) such that for \( n > N \) we have \( \hat{A}_n \subseteq V \). Set \( V_1 = \bigcup_{n=1}^{N} (V \cap \hat{A}_n) \) and \( V_2 = V \setminus V_1 \). Clearly, \( V_1 \) and \( V_2 \) are open. Moreover, it is clear that \( \Psi(V_1) \) is open in \( \hat{A} \). If \( \pi_2 \in V_2 \), put \( J = \{x \in \hat{A}_n: x(n) = 0, n < N\} \). If \( \pi_2 \notin V_2 \), put \( J = J_{\pi_2} \cap \{x \in \hat{A}_n: x(n) = 0, n < N\} \). It is easy to see that in either case \( J \) is a closed two-sided ideal of \( \hat{A} \) and that \( \Psi(V_2) = \hat{A}_J \). So \( \Psi(V) = \Psi(V_1) \cup \Psi(V_2) \) is an open neighborhood of \( \rho_1 \). Similarly \( \Psi(V) \) is an open neighborhood of \( \rho_2 \) whenever \( V \) is an open neighborhood of \( \pi_2 \). This completes the proof.

**Remark.** Let \( \{(B_n, p'_n)\} \) be any rearrangement of the sequence \( \{(A_n, p_n)\} \). The above result clearly implies that the spectrum of \( \hat{A}(\{(A_n, p_n)\}) \) is homeomorphic to the spectrum of \( \hat{A}(\{(B_n, p'_n)\}) \). Actually a stronger statement can be made. The \( C^* \)-algebra \( \hat{A}(\{(A_n, p_n)\}) \) is \( * \)-isomorphic to the \( C^* \)-algebra \( \hat{A}(\{(B_n, p'_n)\}) \). Of course, if each \( A_n \) is separable then, by [2, 9.5.3], each \( A_n \) is liminal since it has a \( T_1 \) spectrum.

**3.2. Proof of the Theorem.** Let \( A_1 \) be the \( C^* \)-algebra of all sequences of \( 2 \times 2 \) matrices of complex numbers which converge to a diagonal matrix. By [2, 4.7.19], \( A_1 \) is a unital separable liminal \( C^* \)-algebra whose spectrum is homeomorphic to the space \( X_1 \) defined in Lemma 2.1. Let \( \gamma \) be a countable ordinal and suppose, for each \( \rho < \gamma \), \( A_\rho \) is a unital separable liminal \( C^* \)-algebra whose spectrum is homeomorphic to the space \( X_\rho \) defined in Lemma 2.1. First, assume \( \gamma \) is not a limit ordinal. For each positive integer \( n \), let \( M_2(A_{\gamma-1}) \) denote the \( C^* \)-algebra of \( 2 \times 2 \) matrices over \( A_{\gamma-1} \). From [3, Corolaires 5, Lemme 16] we see that \( M_2(A_{\gamma-1}) \) is a unital separable liminal \( C^* \)-algebra whose spectrum \( M_2^*(A_{\gamma-1}) \) is homeomorphic to \( \hat{A}_{\gamma-1} \), the spectrum of \( A_{\gamma-1} \). Thus \( M_2(A_{\gamma-1}) \) is homeomorphic to \( X_{\gamma-1} \). For each positive integer \( n \) let \( B_n = M_2(A_{\gamma-1}) \) and
\[
p_n = \begin{bmatrix} I_{\gamma-1} & 0 \\ 0 & 0 \end{bmatrix},
\]
where $I_{y-1}$ is the identity of $A_{y-1}$. Now set $A_y = \mathcal{B}((B_n, p_n))$ as defined in Proposition 3.1. Thus by Proposition 3.1 and our induction hypothesis $A_y$ is a unital separable liminal $C^*$-algebra whose spectrum $\hat{A}_y$ is homeomorphic to $X_y$.

Now assume $y$ is a limit ordinal. Since $y$ is countable, the set of all $\rho < y$ can be put in a sequence $(\rho_n)_{n=1}^\infty$. For each positive integer $n$, let $B_n = M_2(A_{\rho_n})$ and

$$p_n = \begin{bmatrix} I_{\rho_n} & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_{\rho_n}$ is the identity of $A_{\rho_n}$. Set $A_y = \mathcal{B}((B_n, p_n))$ as defined in Proposition 3.1. Again by Proposition 3.1, [3] and our induction hypothesis, $A_y$ is a unital separable liminal $C^*$-algebra whose spectrum $\hat{A}_y$ is homeomorphic to $X_y$. The assertion of the theorem follows immediately from Lemma 2.1 and [2, 3.2.1].

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