THE EFFICIENCY OF AN ALGORITHM
OF INTEGER PROGRAMMING:
A PROBABILISTIC ANALYSIS

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Abstract. A simple algorithm for solving the knapsack problem is shown to lead
to examining, on the average, around $e^{2^{1/8}}$ vectors out of $2^n$.

Most optimization methods can be considered as devices for reducing the
number of possible solutions that need to be examined and compared. The
efficiency of an algorithm is determined to a large extent by what fraction of the
total number of possible solutions has to be examined in the process of application
of the algorithm. Computational experience shows that this fraction is usually very
small for many inputs, but not for all: even the simplex method is shown to be
inefficient in the worst case [1]. Accordingly, such algorithms are relatively fast for
many inputs, but do not have good uniform computing time bounds. Moreover,
most operations research problems are known to be NP-hard [2] and one can
expect any algorithm for any NP-hard problem to be intractable in the sense of
worst case performance.

For this reason, it is of interest to obtain the average values of parameters
determining the computing time. In this paper we compute the expected number of
vectors examined in the course of application of a simple algorithm for the
knapsack problem (zero-one programming problem with a single constraint). The
algorithm is quite primitive as compared to the methods actually used in computa-
tional practice. However, the idea employed in this algorithm to reduce the number
of possible solutions is similar to the ideas on which "real" integer programming
algorithms are based. It is hoped, therefore, that the result below may give a rough
idea of the computational efficiency of "real" methods. The average computing
time of such methods is very hard to find, and one often relies on empirical testing.
Horowitz and Sahni [3] compare the results of an empirical study of a few
estimating the average efficiency of backtracking algorithms.

During this work I have benefited from conversations with many mathemati-
cians, and I should especially mention I. Davidova, N. Maslova, B. Pittel, I.
Romanovsky and the late Yu. Burtin.
1. The knapsack problem may be formulated as follows.

Maximize \( z = \sum_{i=1}^{n} c_i x_i \)

subject to: \( \sum_{i=1}^{n} a_i x_i \leq w, \quad x_i = 0, 1 (i = 1, \ldots, n). \)

Thus, a particular problem from this class is determined by a vector

\[ p = \langle a, c, w \rangle = \langle a_1, \ldots, a_n, c_1, \ldots, c_n, w \rangle. \]

It is possible to rearrange any such problem so that \( c > 0 \) by making the substitution \( x_i = 1 - x'_i \) for \( c_i < 0 \). Furthermore, if \( c > 0 \) and \( a_i < 0 \) for some \( i \) then one may assume \( x_i = 1 \), so that the number of variables can be reduced. If \( a, c > 0, w < 0 \) then the problem is trivial. For these reasons we may assume \( p > 0 \).

For any problem \( p \), define the partial ordering \( \prec_p \) of \( \{1, \ldots, n\} \) as follows.

\( i \prec_p j \) if

(a) \( a_i > a_j, c_i < c_j \) and

(b) \( a_i = a_j, c_i = c_j \) imply \( i > j \).

Let \( B_p \) be the set of all zero-one vectors \( x = \langle x_1, \ldots, x_n \rangle \) such that \( x_i \leq x_j \) for \( i \prec_p j \). If feasible solutions exist then \( B_p \) contains an optimal solution.

The algorithm we consider in this paper consists in generating successively all vectors from \( B_p \), checking whether a generated vector is feasible and comparing the values of the objective function at the feasible points. The efficiency of this procedure for a given problem \( p \) is determined by the number \( |B_p| \) of generated vectors as compared to the total number \( 2^n \) of zero-one vectors. (We do not describe any particular method for generating the elements of \( B_p \) since the choice of a method does not affect the measure of complexity we use.)

Consider a random knapsack problem, i.e. a nonnegative random vector \( p \). What distribution of \( p \) should be assumed? The actual distribution in practical situations may depend on the particular (extramathematical) source of knapsack problems. A priori, no distribution seems very convincing, but two situations are especially interesting [3]: the case when greater components of \( a \) correspond to greater components of \( c \) and the case when all components are independent. In the first case, the method under consideration does not give any economy: obviously, \( |B_p| = 2^n \). In the second case, assuming additionally that the components of \( a, c \) are distributed identically, and their distribution is continuous, we have the following result.

**Theorem.** \( E|B_p| = \sum_{k=0}^{n} (k!)^{-1} \binom{n}{k}. \)

**Corollary.** \( \ln E|B_p| \sim 2\sqrt{n} \).

Thus the expected number of generated vectors is about \( e^{2\sqrt{n}} \) out of \( 2^n \).

2. To prove the Theorem, consider the following combinatorial problem about the permutations of \( \{1, \ldots, n\} \). An inversion in a permutation \( \alpha \) is an arbitrary pair of indices \( \langle i, j \rangle \) such that \( i < j, \alpha_i > \alpha_j \). A subset \( X \) of \( \{1, \ldots, n\} \) is \( \alpha \)-closed
if, for every inversion \( \langle i, j \rangle \) in \( \alpha \), \( i \in X \) implies \( j \in X \). Let \( g_{\alpha} \) be the number of \( \alpha \)-closed sets. What is the expectation of \( g_{\alpha} \) for a uniformly distributed \( \alpha \)?

Let \( p \) be a problem with \( a_i \neq a_j, c_i \neq c_j \) for \( i \neq j \). Let \( \alpha^p \) be the number of elements of \( \{a_1, \ldots, a_n\} \) not greater than \( a_i \), \( \gamma^p \) be the number of elements of \( \{c_1, \ldots, c_n\} \) not greater than \( c_i \). Obviously, \( \alpha^p, \gamma^p \) are permutations. For all \( \alpha, \gamma \) define

\[
\Pi_{\alpha, \gamma} = \{ p : \alpha^p = \alpha, \gamma^p = \gamma \}.
\]

**Lemma 1.** If \( p \in \Pi_{\alpha, \gamma} \) then \(|B_p| = g_{\gamma \alpha^{-1}}\).

**Proof.**

\[
i < j \iff a_i > a_j \text{ and } c_i < c_j
\]

\[
\iff \alpha^p > \alpha^p \text{ and } \gamma^p < \gamma^p
\]

\[
\iff \alpha_i > \alpha_j \text{ and } \gamma_i < \gamma_j
\]

\[
\iff \alpha_i > \alpha_j \text{ and } (\gamma \alpha^{-1})_a_i < (\gamma \alpha^{-1})_a_j
\]

\[
\iff \langle \alpha_i, \alpha_j \rangle \text{ is an inversion in } \gamma \alpha^{-1}.
\]

\[
x \in B_p \iff \forall i,j (i < j \implies x_i < x_j)
\]

\[
\iff \forall i,j ((i < j \text{ and } x_j = 0) \implies x_i = 0)
\]

\[
\iff \forall i,j ((\langle a_i, a_j \rangle \text{ is an inversion in } \gamma \alpha^{-1} \text{ and } x_j = 0) \implies x_i = 0)
\]

\[
\iff [a_i : x_i = 0] \text{ is } \gamma \alpha^{-1} \text{-closed.}
\]

Hence the mapping \( x \mapsto [a_i : x_i = 0] \) maps \( B_p \) onto the set of \( \gamma \alpha^{-1} \)-closed sets. This mapping is obviously a bijection.

**Lemma 2.** For a uniformly distributed \( \beta \), \( g_{\beta} \) has the same distribution as \(|B_p|\).

**Proof.** The events \( \Pi_{\alpha, \gamma} \) are disjoint, have equal probabilities, and their union is the sure event. Hence

\[
P[p \in \Pi_{\alpha, \gamma}] = (n!)^{-2}.
\]

From this fact and Lemma 1,

\[
P[|B_p| = t] = \sum_{\alpha, \gamma} P[|B_p| = t, p \in \Pi_{\alpha, \gamma}]
\]

\[
= \sum_{\alpha, \gamma} P[g_{\gamma \alpha^{-1}} = t, p \in \Pi_{\alpha, \gamma}]
\]

\[
= \sum_{\alpha, \gamma} P[p \in \Pi_{\alpha, \gamma}] \sum_{\beta, \gamma \alpha^{-1} = t} (n!)^{-2}
\]

\[
= \sum_{\beta : g_{\beta} = t} (n!)^{-1} = P[g_{\beta} = t].
\]

3. To find \( E_{g_{\beta}} \), define, for any permutation \( \beta \) of \( \{1, \ldots, n\} \), permutations \( \beta', \beta'' \) of initial segments of \( \{1, \ldots, n\} \) as follows. Let \( \beta = (\beta_1, \ldots, \beta_i, \ldots, \beta_n) \), where
\(\beta_i = n\). Then define
\[
\beta' = (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n), \quad \beta'' = (\tilde\beta_1, \ldots, \tilde\beta_{i-1}),
\]
where \(\tilde\beta_v (1 < v < i)\) is the number of the elements of \(\{\beta_1, \ldots, \beta_{i-1}\}\) which are < \(v\). \(\beta', \beta''\) are permutations of \(\{1, \ldots, n-1\}\) and \(\{1, \ldots, i-1\}\), respectively.

**Example.** For \(\beta = (2, 4, 5, 1, 3)\), \(i = 3\), \(\beta' = (2, 4, 1, 3)\), \(\beta'' = (1, 2)\).

**Lemma 3.** \(g_\beta = g_{\beta'} + g_{\beta''}\).

**Proof.** The set of \(\beta\)-closed sets that do not include \(i\) is in the obvious 1-1 correspondence with the set of \(\beta'\)-closed sets. Any \(\beta\)-closed set including \(i\) includes also \(i+1, \ldots, n\). Hence there is a 1-1 correspondence between such sets and the \(\beta''\)-closed sets.

Define \(f_n = Eg_\beta\).

**Lemma 4.**
\[
f_0 = 1, \quad f_n = f_{n-1} + n^{-1} \sum_{s=0}^{n-1} f_s \quad (n > 0).
\]

**Proof.** The first formula is obvious. Lemma 3 implies \(Eg_\beta = Eg_{\beta'} + Eg_{\beta''}\). The left-hand side is \(f_n\). Moreover, \(\beta'\) is the uniformly distributed permutation of \(\{1, \ldots, n-1\}\). Hence the first summand in the right-hand side is \(f_{n-1}\). \(\beta''\) is a permutation of \(\{1, \ldots, i-1\}\), where \(i\) is random itself; it is equal to \(1, \ldots, n\) with equal probabilities. For any \(s = 1, \ldots, n\), \(\beta''\) is the uniformly distributed permutation of \(\{1, \ldots, s-1\}\) provided \(i = s\). Hence the second summand is \(n^{-1}\sum_{s=0}^{n} f_s\).

**Lemma 5.** \(f_n = \sum_{k=0}^{n-1} (k!)^{-1} \binom{n}{k}\).

**Proof.** We use induction on \(n\) and Lemma 4. The basis is trivial. The induction step:
\[
f_n = f_{n-1} + \frac{1}{n} \sum_{s=0}^{n-1} f_s = f_{n-1} + \frac{1}{n} \sum_{s=0}^{n-1} \sum_{k=0}^{s} (k!)^{-1} \binom{s}{k}
\]
\[
= f_{n-1} + \frac{1}{n} \sum_{k=0}^{n-1} (k!)^{-1} \sum_{s=k}^{n-1} \binom{s}{k} = f_{n-1} + \frac{1}{n} \sum_{k=0}^{n-1} (k!)^{-1} \binom{n}{k+1}
\]
\[
= f_{n-1} + \sum_{k=0}^{n-1} \frac{1}{n} \frac{1}{n} (k+1)! \binom{n}{k+1} = f_{n-1} + \sum_{k=0}^{n-1} [(k+1)!]^{-1} \binom{n}{k}
\]
\[
= f_{n-1} + \sum_{k=1}^{n} (k!)^{-1} \binom{n-1}{k-1}
\]
\[
= \sum_{k=0}^{n-1} (k!)^{-1} \binom{n-1}{k} + \sum_{k=1}^{n} (k!)^{-1} \binom{n-1}{k-1}
\]
\[
= 1 + \sum_{k=1}^{n} (k!)^{-1} \left[ \binom{n-1}{k} + \binom{n-1}{k-1} \right] + (n!)^{-1}
\]
\[
= 1 + \sum_{k=1}^{n} (k!)^{-1} \binom{n}{k} + (n!)^{-1} = \sum_{k=0}^{n} (k!)^{-1} \binom{n}{k}.
\]
The Theorem follows from Lemmas 2 and 5.

4. Let $z_k = \binom{n}{k}$, $n > 0$, $0 < k < n$. The ratio $z_{k+1}/z_k = (n - k)/(k + 1)^2$ decreases when $k$ increases. Hence for some $k$,

$$z_1/z_0 > \cdots > z_k/z_{k-1} > 1 > (z_{k+1}/z_k) > \cdots > z_n/z_{n-1}.$$  

Then $z_k$ is the maximum summand of $f_n = \sum_{k=0}^n z_k$ so that $z_k < f_n < (n + 1)z_k$. An easy estimate of $z_{k+1}/z_k$ shows that $\sqrt{n} - 3 < k < \sqrt{n}$. Using Stirling’s approximation, we conclude that

$$\ln f_n = \ln z_k + \mathcal{O}(\ln n)$$

$$= \ln(n!) - \ln[(n - k)!] - 2 \ln(k!) + \mathcal{O}(\ln n)$$

$$= \ln(n!) - \ln[(n - \sqrt{n})!] - 2 \ln(\sqrt{n}!) + \mathcal{O}(\ln n)$$

$$= [n \ln n - n] - [(n - \sqrt{n})\ln(n - \sqrt{n}) - (n - \sqrt{n})]$$

$$- 2[\sqrt{n} \ln(\sqrt{n}) - \sqrt{n}] + \mathcal{O}(\ln n)$$

$$= n \ln n - n \ln(n - \sqrt{n}) + \sqrt{n} \ln(n - \sqrt{n}) - \sqrt{n} \ln n + \sqrt{n} + \mathcal{O}(\ln n)$$

$$= (\sqrt{n} - n)\ln(1 - 1/\sqrt{n}) + \sqrt{n} + \mathcal{O}(\ln n)$$

$$= (\sqrt{n} - n)(-1/\sqrt{n} + \mathcal{O}(1/\sqrt{n})) + \sqrt{n} + \mathcal{O}(\ln n)$$

$$= 2\sqrt{n} + \mathcal{O}(\sqrt{n}).$$

REFERENCES


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