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THE EFFICIENCY OF AN ALGORITHM OF INTEGER PROGRAMMING: A PROBABILISTIC ANALYSIS

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ABSTRACT. A simple algorithm for solving the knapsack problem is shown to lead to examining, on the average, around $e^{2\sqrt{n}}$ vectors out of 2^n.

Most optimization methods can be considered as devices for reducing the number of possible solutions that need to be examined and compared. The efficiency of an algorithm is determined to a large extent by what fraction of the total number of possible solutions has to be examined in the process of application of the algorithm. Computational experience shows that this fraction is usually very small for many inputs, but not for all: even the simplex method is shown to be inefficient in the worst case [1]. Accordingly, such algorithms are relatively fast for many inputs, but do not have good uniform computing time bounds. Moreover, most operations research problems are known to be NP-hard [2] and one can expect any algorithm for any NP-hard problem to be intractable in the sense of worst case performance.

For this reason, it is of interest to obtain the average values of parameters determining the computing time. In this paper we compute the expected number of vectors examined in the course of application of a simple algorithm for the knapsack problem (zero-one programming problem with a single constraint). The algorithm is quite primitive as compared to the methods actually used in computational practice. However, the idea employed in this algorithm to reduce the number of possible solutions is similar to the ideas on which "real" integer programming algorithms are based. It is hoped, therefore, that the result below may give a rough idea of the computational efficiency of "real" methods. The average computing time of such methods is very hard to find, and one often relies on empirical testing. Horowitz and Sahni [3] compare the results of an empirical study of a few knapsack algorithms. Knuth [4] discusses a general Monte-Carlo method for estimating the average efficiency of backtracking algorithms.

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1. The knapsack problem may be formulated as follows.

$$\text{Maximize } z = \sum_{i=1}^{n} c_i x_i$$

$$\text{subject to: } \sum_{i=1}^{n} a_i x_i \leqslant w, \qquad x_i = 0, 1 \ (i = 1, \ldots, n).$$

Thus, a particular problem from this class is determined by a vector

$$p = \langle a, c, w \rangle = \langle a_1, \ldots, a_n, c_1, \ldots, c_n, w \rangle.$$

It is possible to rearrange any such problem so that $c \geqslant 0$ by making the substitution $x_i = 1 - x_i'$ for $c_i < 0$. Furthermore, if $c \geqslant 0$ and $a_i < 0$ for some i then one may assume $x_i = 1$, so that the number of variables can be reduced. If $a, c \geqslant 0$, $w < 0$ then the problem is trivial. For these reasons we may assume $p \geqslant 0$.

For any problem p, define the partial ordering $<_p$ of $\{1, \ldots, n\}$ as follows. $i <_p j$ if

(a) $a_i \geqslant a_j$, $c_i \leqslant c_j$ and

(b) $a_i = a_j$, $c_i = c_j$ imply $i > j$.

Let B_p be the set of all zero-one vectors $x = \langle x_1, \ldots, x_n \rangle$ such that $x_i \leqslant x_j$ for $i <_p j$. If feasible solutions exist then B_p contains an optimal solution.

The algorithm we consider in this paper consists in generating successively all vectors from B_p, checking whether a generated vector is feasible and comparing the values of the objective function at the feasible points. The efficiency of this procedure for a given problem p is determined by the number $|B_p|$ of generated vectors as compared to the total number 2^n of zero-one vectors. (We do not describe any particular method for generating the elements of B_p since the choice of a method does not affect the measure of complexity we use.)

Consider a random knapsack problem, i.e. a nonnegative random vector p. What distribution of p should be assumed? The actual distribution in practical situations may depend on the particular (extramathematical) source of knapsack problems. A priori, no distribution seems very convincing, but two situations are especially interesting [3]: the case when greater components of a correspond to greater components of c and the case when all components are independent. In the first case, the method under consideration does not give any economy: obviously, $|B_p| = 2^n$. In the second case, assuming additionally that the components of a, c are distributed identically, and their distribution is continuous, we have the following result.

THEOREM. $E|B_p| = \sum_{k=0}^{n} (k!)^{-1} \binom{n}{k}$.

COROLLARY. $\ln E|B_p| \sim 2\sqrt{n}$.

Thus the expected number of generated vectors is about $e^{2\sqrt{n}}$ out of 2^n.

2. To prove the Theorem, consider the following combinatorial problem about the permutations of $\{1, \ldots, n\}$. An *inversion* in a permutation α is an arbitrary pair of indices $\langle i, j \rangle$ such that $i < j$, $\alpha_i > \alpha_j$. A subset X of $\{1, \ldots, n\}$ is *α-closed* if, for every inversion $\langle i, j\rangle$ in α, $i \in X$ implies $j \in X$. Let g_α be the number of α-closed sets. What is the expectation of g_α for a uniformly distributed α?

Let p be a problem with $a_i \neq a_j$, $c_i \neq c_j$ for $i \neq j$. Let α_i^p be the number of elements of $\{a_1, \ldots, a_n\}$ not greater than a_i, γ_i^p be the number of elements of $\{c_1, \ldots, c_n\}$ not greater than c_i. Obviously, α^p, γ^p are permutations. For all α, γ define

$$\Pi_{\alpha,\gamma} = \{p: \alpha^p = \alpha, \gamma^p = \gamma\}.$$

LEMMA 1. *If $p \in \Pi_{\alpha,\gamma}$ then $|B_p| = g_{\gamma\alpha^{-1}}$.*

PROOF.

$$\begin{aligned} i \underset{p}{<} j &\Leftrightarrow a_i > a_j \text{ and } c_i < c_j \\ &\Leftrightarrow \alpha_i^p > \alpha_j^p \text{ and } \gamma_i^p < \gamma_j^p \\ &\Leftrightarrow \alpha_i > \alpha_j \text{ and } \gamma_i < \gamma_j \\ &\Leftrightarrow \alpha_i > \alpha_j \text{ and } (\gamma\alpha^{-1})_{\alpha_i} < (\gamma\alpha^{-1})_{\alpha_j} \\ &\Leftrightarrow \langle \alpha_j, \alpha_i\rangle \text{ is an inversion in } \gamma\alpha^{-1}; \end{aligned}$$

$$\begin{aligned} x \in B_p &\Leftrightarrow \forall ij\big(i \underset{p}{<} j \Rightarrow x_i \leqslant x_j\big) \\ &\Leftrightarrow \forall ij\Big(\big(i \underset{p}{<} j \text{ and } x_j = 0\big) \Rightarrow x_i = 0\Big) \\ &\Leftrightarrow \forall ij\Big(\big(\langle \alpha_j, \alpha_i\rangle \text{ is an inversion in } \gamma\alpha^{-1} \text{ and } x_j = 0\big) \Rightarrow x_i = 0\Big) \\ &\Leftrightarrow \big[\alpha_i \colon x_i = 0\big] \text{ is } \gamma\alpha^{-1}\text{-closed}. \end{aligned}$$

Hence the mapping $x \mapsto [\alpha_i\colon x_i = 0]$ maps B_p onto the set of $\gamma\alpha^{-1}$-closed sets. This mapping is obviously a bijection.

LEMMA 2. *For a uniformly distributed β, g_β has the same distribution as $|B_p|$.*

PROOF. The events $\Pi_{\alpha,\gamma}$ are disjoint, have equal probabilities, and their union is the sure event. Hence

$$P\big[p \in \Pi_{\alpha,\gamma}\big] = (n!)^{-2}.$$

From this fact and Lemma 1,

$$\begin{aligned} P\big[|B_p| = t\big] &= \sum_{\alpha,\gamma} P\big[|B_p| = t, p \in \Pi_{\alpha,\gamma}\big] \\ &= \sum_{\alpha,\gamma} P\big[g_{\gamma\alpha^{-1}} = t, p \in \Pi_{\alpha,\gamma}\big] \\ &= \sum_{\substack{\alpha,\gamma \\ g_{\gamma\alpha^{-1}} = t}} P\big[p \in \Pi_{\alpha,\gamma}\big] = \sum_{\substack{\beta \\ g_\beta = t}} \sum_{\substack{\gamma,\alpha \\ \gamma\alpha^{-1} = \beta}} (n!)^{-2} \\ &= \sum_{\beta; g_\beta = t} (n!)^{-1} = P\big[g_\beta = t\big]. \end{aligned}$$

3. To find Eg_β, define, for any permutation β of $\{1, \ldots, n\}$, permutations β', β'' of initial segments of $\{1, \ldots, n\}$ as follows. Let $\beta = (\beta_1, \ldots, \beta_i, \ldots, \beta_n)$, where

$\beta_i = n$. Then define

$$\beta' = (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n), \qquad \beta'' = (\tilde{\beta}_1, \ldots, \tilde{\beta}_{i-1}),$$

where $\tilde{\beta}_v$ $(1 \leqslant v < i)$ is the number of the elements of $\{\beta_1, \ldots, \beta_{i-1}\}$ which are $\leqslant v$. β', β'' are permutations of $\{1, \ldots, n-1\}$ and $\{1, \ldots, i-1\}$, respectively.

EXAMPLE. For $\beta = (2, 4, 5, 1, 3)$, $i = 3$, $\beta' = (2, 4, 1, 3)$, $\beta'' = (1, 2)$.

LEMMA 3. $g_\beta = g_{\beta'} + g_{\beta''}$.

PROOF. The set of β-closed sets that do not include i is in the obvious 1-1 correspondence with the set of β'-closed sets. Any β-closed set including i includes also $i+1, \ldots, n$. Hence there is a 1-1 correspondence between such sets and the β''-closed sets.

Define $f_n = Eg_\beta$.

LEMMA 4.

$$f_0 = 1, \qquad f_n = f_{n-1} + n^{-1}\sum_{s=0}^{n-1} f_s \quad (n > 0).$$

PROOF. The first formula is obvious. Lemma 3 implies $Eg_\beta = Eg_{\beta'} + Eg_{\beta''}$. The left-hand side is f_n. Moreover, β' is the uniformly distributed permutation of $\{1, \ldots, n-1\}$. Hence the first summand in the right-hand side is f_{n-1}. β'' is a permutation of $\{1, \ldots, i-1\}$, where i is random itself; it is equal to $1, \ldots, n$ with equal probabilities. For any $s = 1, \ldots, n$, β'' is the uniformly distributed permutation of $\{1, \ldots, s-1\}$ provided $i = s$. Hence the second summand is $n^{-1}\sum_{s=0}^{n} f_s$.

LEMMA 5. $f_n = \sum_{k=0}^{n-1}(k!)^{-1}\binom{n}{k}$.

PROOF. We use induction on n and Lemma 4. The basis is trivial. The induction step:

$$\begin{aligned}
f_n &= f_{n-1} + \frac{1}{n}\sum_{s=0}^{n-1} f_s = f_{n-1} + \frac{1}{n}\sum_{s=0}^{n-1}\sum_{k=0}^{s}(k!)^{-1}\binom{s}{k} \\
&= f_{n-1} + \frac{1}{n}\sum_{k=0}^{n-1}(k!)^{-1}\sum_{s=k}^{n-1}\binom{s}{k} = f_{n-1} + \frac{1}{n}\sum_{k=0}^{n-1}(k!)^{-1}\binom{n}{k+1} \\
&= f_{n-1} + \sum_{k=0}^{n-1}[(k+1)!]^{-1}\frac{k+1}{n}\binom{n}{k+1} = f_{n-1} + \sum_{k=0}^{n-1}[(k+1)!]^{-1}\binom{n-1}{k} \\
&= f_{n-1} + \sum_{k=1}^{n}(k!)^{-1}\binom{n-1}{k-1} \\
&= \sum_{k=0}^{n-1}(k!)^{-1}\binom{n-1}{k} + \sum_{k=1}^{n}(k!)^{-1}\binom{n-1}{k-1} \\
&= 1 + \sum_{k=1}^{n-1}(k!)^{-1}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] + (n!)^{-1} \\
&= 1 + \sum_{k=1}^{n-1}(k!)^{-1}\binom{n}{k} + (n!)^{-1} = \sum_{k=0}^{n}(k!)^{-1}\binom{n}{k}.
\end{aligned}$$

The Theorem follows from Lemmas 2 and 5.

4. Let $z_k = (k!)^{-1}\binom{n}{k}$, $n > 0$, $0 \leqslant k \leqslant n$. The ratio $z_{k+1}/z_k = (n-k)/(k+1)^2$ decreases when k increases. Hence for some $\bar{k}$,

$$z_1/z_0 > \cdots > z_{\bar{k}}/z_{\bar{k}-1} \geqslant 1 > (z_{\bar{k}+1})/z_{\bar{k}} > \cdots > z_n/z_{n-1}.$$

Then $z_{\bar{k}}$ is the maximum summand of $f_n = \sum_{k=0}^{n} z_k$ so that $z_{\bar{k}} < f_n < (n+1)z_{\bar{k}}$. An easy estimate of z_{k+1}/z_k shows that $\sqrt{n} - 3 < \bar{k} < \sqrt{n}$. Using Stirling's approximation, we conclude that

$$\begin{aligned}
\ln f_n &= \ln z_{\bar{k}} + O(\ln n) \\
&= \ln(n!) - \ln\left[(n-\bar{k})!\right] - 2\ln(\bar{k}!) + O(\ln n) \\
&= \ln(n!) - \ln\left[(n-\sqrt{n})!\right] - 2\ln(\sqrt{n}\,!) + O(\ln n) \\
&= \left[n \ln n - n\right] - \left[(n-\sqrt{n})\ln(n-\sqrt{n}) - (n-\sqrt{n})\right] \\
&\quad -2\left[\sqrt{n}\,\ln(\sqrt{n}) - \sqrt{n}\,\right] + O(\ln n) \\
&= n\ln n - n\ln(n-\sqrt{n}) + \sqrt{n}\,\ln(n-\sqrt{n}) - \sqrt{n}\,\ln n + \sqrt{n} + O(\ln n) \\
&= (\sqrt{n} - n)\ln(1 - 1/\sqrt{n}) + \sqrt{n} + O(\ln n) \\
&= (\sqrt{n} - n)\left(-1/\sqrt{n} + O(1/\sqrt{n})\right) + \sqrt{n} + O(\ln n) \\
&= 2\sqrt{n} + O(\sqrt{n}).
\end{aligned}$$

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