TRIANGLES IN ARRANGEMENTS OF LINES. II

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Abstract. We show that given n lines in the real projective plane, no n — 1 of which are concurrent, the number $p_3$ of triangular regions formed is at most $\frac{2}{3}n(n-1)$, equality being possible.

We also show that if $n > 6$ then $p_3 < \frac{7}{18}n(n-1) + \frac{1}{3}$. Grünbaum has conjectured $p_3 < \frac{1}{3}n(n-1)$.

1. Introduction. Let $A$ be an arrangement of $n$ lines in the real projective plane $P_2$, and let $p_k$ denote the number of $k$-sided regions determined by the lines. It is assumed that $A$ is not a pencil. In [1] Grünbaum raises the question of finding an upper bound for $p_3$ and conjectures that

$$p_3 < \left[\frac{1}{3}n\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)\right] < \frac{1}{3}n(n-1)$$

for $n > 16$, and gives R. J. Canham's proof of this when $A$ is simple (no three lines concurrent). The first nontrivial results on the general problem were found by T. O. Strommer [3] who proved among other things that

$$p_3 < \frac{1}{3}n(n-1) + 2 + \frac{1}{3}\sum_{k>4}(k-6)p_k.$$

In [2] we extended these results and obtained some inequalities that implied $p_3 < \frac{5}{12}n(n-1)$. We now extend these results further, proving two results.

**Theorem 1.** $p_3 < \frac{3}{2}n(n-1)$ if $A$ is not a near pencil ($n-1$ lines concurrent), and the result is best possible since equality occurs when $n = 6$.

**Theorem 2.** $p_3 < \frac{7}{18}n(n-1) + \frac{1}{3}$ if $A$ is not a near pencil.

**Remark.** If $A$ is a near pencil, then $p_3 = 2n - 2$.

2. The proofs.

Proof of Theorem 1. Let $A$ be an arrangement of $n$ lines in $P_2$, not a pencil or near pencil. Let $t_k$ be the number of points incident with precisely $k$ lines. For a fixed line $l$ of the arrangement, let $t_k(l)$ denote the number of points on $l$ incident with precisely $k$ lines, including $l$. Let us call a point incident with precisely two lines a simple point, and the other points multiple points. Let $P_1, \ldots, P_m$ be the multiple points, in order, on the line $l$, with the understanding that $m = 0$ if there are no multiple points on $l$. We let $I_i = (P_i, P_{i+1})$ called the $i$th block be the set of simple points on $l$ between $P_i$ and $P_{i+1}$ on $l$, with the understanding that

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(P_m', P_{m+1}) = (P_m, P_1). In the event that m = 1 we have the single block (P_1, P_1). We must therefore allow for the possibility that the endpoints of I_i = (P_i, P_{i+1}) are the same point. If m = 0 there are no blocks. Each block I_i determines |I_i| + 1 line segments \( \overline{P_iQ_1}, \overline{Q_1Q_2}, \ldots, \overline{Q_{k-1}Q_k}, \overline{Q_kP_{i+1}} \), where I_i = \( \{Q_1, \ldots, Q_k\} \), where |S| denotes the cardinality of the set S. If I_i = \( \emptyset \), then there is only the line segment \( \overline{P_iP_{i+1}} \). Let \( 1 < j < k \). The line segment \( \overline{Q_jQ_{j+1}} \) is the side of at most one triangle, for suppose \( \overline{Q_jQ_{j+1}} \) is incident with two triangles. Then the removal of I_i would result in a digon (a two-sided polygon), and digons only occur in pencils, contradicting the assumption that \( A \) is not a near pencil. Thus, if \( p_3(I) \) denotes the number of triangles having a side lying on I, then the contribution of I_i to \( p_3(I) \) is at most \( k + 3 = |I_i| + 3 \). If in fact the contribution is less and if I_i \( \neq \emptyset \), then we say that I_i is a deficient block. Let \( D_I \) denote the number of deficient blocks on I and \( E_I \) the number of empty blocks on I. If I_i = \( \emptyset \), then the contribution to \( p_3(I) \) is only \( |I_i| + 2 = 2 \). Hence, if \( m > 1 \),

\[
p_3(I) < \sum_i \{ |I_i| + 3 \} - D_I - E_I
\]

If \( m = 0 \),

\[
p_3(I) < t_2(I) = t_2(l) + 3 \sum_{k \geq 3} t_k(I) - D_I - E_I,
\]

and so, in any case,

\[
p_3(I) < t_2(l) + 3 \sum_{k \geq 3} t_k(I) - D_I - E_I.
\]

If we let E and D denote respectively the number of empty and deficient blocks on all lines, then taking into account that triangles get counted three times and points on k lines get counted k times, we obtain on summation over I,

\[
3p_3 < 2t_2 + 3 \sum_{k \geq 3} kt_k - E - D.
\]

We need to consider so-called F-blocks. Let |I_i| = 2, I_i = \( \{Q_1, Q_2\} \), and suppose that I_i is not deficient. Then we have the situation depicted in Figure 1, with |I_i| + 3 = 5 triangles numbered from one to five. We call such a block a type F-block and it forces the existence of a point P incident with at least four lines. Let us give the segments \( \overline{Q_1P} \) and \( \overline{Q_2P} \) weight one to indicate that they cannot occur in any other block, and give \( \overline{P_iP} \) and \( \overline{P_{i+1}P} \) weight \( \frac{1}{2} \) to indicate that they can each occur in at most two blocks (or possibly \( P_i = P_{i+1} \)). The sum of these weights is \( 3 \sum F' = 3F \), where \( F' \) is the number of F-blocks on I, and F is the total number of F-blocks. The sum of these weights cannot exceed \( \sum_{k \geq 4} 2kt_k \), and so we have

\[
\frac{4}{3} \sum_{k \geq 4} kt_k \geq 2F.
\]

Let us now look at a typical block I_i which is not an empty block, a deficient block or an F-block. I_i = \( \{Q_1, \ldots, Q_k\} \). See Figure 2. The triangles are marked with the symbol \( \Delta \). The points P and P' are each incident with three or more lines,
and we have given $P_i P$ and $P_{i+1} P'$ weight $\frac{1}{2}$, and we have given $Q_1 P$, $Q_2 P$, $Q_{k-1} P'$ and $Q_k P'$ weight one, and block $I_i$ has total weight 5 in the figure drawn. In fact, $P$ and $P'$ could lie on the same side of $l$ and might even coincide. We claim that in all cases the total weight will be at least 4. In Figure 3 we see how 4 can occur. If $P_i$ and $P_{i+1}$ coincide it is to be understood that each occurrence of $P_i P$ in Figure 2 or 3 gets weight $\frac{1}{2}$, so that the single edge gets, in fact, a weight of one.
The total weight from all of these blocks cannot exceed $\sum_{k \geq 3} 2kt_k$, and the number of these blocks on the line $l$ is

$$\sum_{k \geq 3} t_k(l) - F^l - E^l - D^l.$$  

Hence

$$\sum_{k \geq 3} 2kt_k > 4 \sum_l \left\{ \sum_{k \geq 3} t_k(l) - F^l - E^l - D^l \right\}$$

$$= 4 \left\{ \sum_{k \geq 3} kt_k - F - E - D \right\},$$

and we have

$$2E + 2D > \sum_{k \geq 3} kt_k - 2F,$$

and adding inequality (2), we obtain

$$2E + 2D > 3t_3 - \frac{1}{3} \sum_{k \geq 4} kt_k.$$  

Combining with (1), we get

$$3p_3 < 2t_2 + \frac{15}{2} t_3 + \frac{19}{6} \sum_{k \geq 4} kt_k.$$  

(3)

Let $p_k$ be the number of $k$-gons determined by the region. Counting all of the edges, we see that $\sum kp_k = 2 \sum kt_k$, which we rewrite as

$$3p_3 < 4t_2 + 6t_3 + 2 \sum_{k \geq 4} kt_k - \sum_{k \geq 4} kp_k.$$  

(4)

Adding $\frac{4}{3}$ of (3) to $\frac{1}{3}$ of (4) we obtain

$$3p_3 < \frac{12}{5} t_2 + \frac{36}{5} t_3 + \frac{44}{15} \sum_{k \geq 4} kt_k$$

$$< \frac{12}{5} \left( t_2 + 3t_3 + \sum_{k \geq 4} \binom{k}{2} t_k \right) = \frac{6}{5} n(n - 1),$$

and $p_3 < \frac{3}{4} n(n - 1)$. To see that equality can occur for $n = 6$, see Figure 4, and Theorem 1 is proved.

**Proof of Theorem 2.** Thomas O. Strommer [3] proved

$$p_3 < \frac{1}{3} n(n - 1) + 4 - \frac{2}{3} t_2 + \sum_{k \geq 4} (k - 4)p_k.$$  

(5)

Adding this to (4) and multiplying the result by $\frac{3}{4}$ we obtain

$$3p_3 < \frac{1}{4} n(n - 1) + 3 + \frac{5}{2} t_2 + \frac{9}{2} t_3 + \frac{3}{2} \sum_{k \geq 4} kt_k - 3 \sum_{k \geq 4} p_k.$$  

(6)
If we now add \( \frac{1}{3} \) of (6) to \( \frac{2}{3} \) of (3) we obtain

\[
3p_3 < \frac{1}{12} n(n - 1) + 1 + \frac{13}{6} t_2 + \frac{39}{6} t_3 + \frac{47}{18} \sum_{k \geq 4} kt_k
\]

\[
< \frac{1}{12} n(n - 1) + 1 + \frac{13}{6} \left( t_2 + 3t_3 + \sum_{k \geq 4} \binom{k}{2} t_k \right)
\]

\[
= \frac{14}{12} n(n - 1) + 1,
\]

and Theorem 2 follows.

**Remark.** Inequality (6) gives a rather curious upper bound on \( \sum_{k \geq 3} p_k \), the total number of regions.

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**References**