PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, May 1980

ON THE ITERATIONS OF DIFFEOMORPHISMS WITHOUT C^0-Ω-EXPLOSIONS: AN EXAMPLE

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ABSTRACT. In this note we construct a diffeomorphism f such that f has no C^0-Ω-explosion but f^2 has Ω-explosion.

The purpose of this note is to give an example of a diffeomorphism f on a 2-sphere S^2 such that f has no C^0-Ω-explosion but f^2 has Ω-explosion. As noted below, this is accomplished by finding a diffeomorphism f without C^0-Ω-explosion for which $\Omega(f) \neq \Omega(f^2)$.

Before proceeding to construct the example, we shall make a few observations on the iterations of diffeomorphisms without C^0-Ω-explosions:

A point $x \in S^2$ is said to be a chain recurrent point of f if for any $\varepsilon > 0$ there exists a sequence $\{x_0, \ldots, x_n\}$ of points on S^2 with $x_0 = x_n = x$ and $d(f(x_i), x_{i+1}) < \varepsilon$ where d is a metric on S^2 (cf. [1]). We denote the sets of chain recurrent points and nonwandering points of f by $\mathfrak{R}(f)$ and $\Omega(f)$ respectively. By definitions, it follows that $\Omega(f^m) \subset \Omega(f) \subset \mathfrak{R}(f) = \mathfrak{R}(f^m)$, $m \neq 0$. In [2, Theorem 3.11] M. Shub showed that f has no C^0-Ω-explosion if and only if $\mathfrak{R}(f) = \Omega(f)$. From the above facts, we have

PROPOSITION. *A diffeomorphism f^m has no C^0-Ω-explosion if and only if f has no C^0-Ω-explosion and $\Omega(f) = \Omega(f^m)$.*

Hence for our purpose, it is sufficient to construct a diffeomorphism f on S^2 such that f has no C^0-Ω-explosion and $\Omega(f) \neq \Omega(f^2)$, i.e., $\mathfrak{R}(f) = \Omega(f) \neq \Omega(f^2)$.

The author wishes to thank Professor H. Noguchi, Y. Togawa and K. Yano for helpful comments to this note.

The construction. At first we take a diffeomorphism f' on S^2 to be the time one map of the flow ψ_t on S^2 as pictured in Figures 1 and 2. Here Figures 1 and 2 show the flow ψ_t on the upper and the lower hemispheres, H^+ and H^- respectively, and S^1 is the equator, 0 and 0′ are the north and the south poles respectively, and A' is the antipodal point of A.

For f', there are two fixed points A and A', two fixed sources 0 and 0′, and clearly S^1 is an invariant set. Furthermore let f' satisfy that $|\theta(f'(x))| < |\theta(x)|$ for any $x \in S^2 - (S^1 \cup 0 \cup 0')$ where $\theta(x)$ is the latitude of x $(-\pi/2 < \theta(x) < \pi/2)$ and $|(\ldots)|$ is the absolute value of $(\ldots)$.

Received by the editors February 12, 1979 and, in revised form, May 16, 1979.
AMS (MOS) subject classifications (1970). Primary 58F99; Secondary 57D50.
Key words and phrases. Diffeomorphisms, no C^0-Ω-explosion.

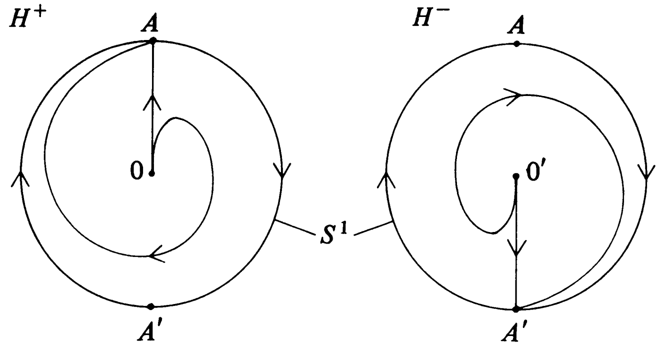

FIGURE 1 FIGURE 2

Now we construct a diffeomorphism f on S^2 as follows; $f = \rho \circ f'$ where ρ is the map such that $\rho(x)$ is the antipodal point of x for any $x \in S^2$. Note that A, A' are periodic points, 0, 0′ are periodic sources of period 2 of f, and that f satisfies

$$|\theta(f(x))| < |\theta(x)| \quad \text{for any } x \in S^2 - (S^1 \cup 0 \cup 0'). \tag{*}$$

LEMMA. $\mathcal{R}(f) = \Omega(f) \neq \Omega(f^2)$.

PROOF. We first show that $\mathcal{R}(f) = 0 \cup 0' \cup S^1$. Clearly $0 \cup 0' \cup S^1 \subset \mathcal{R}(f)$. Hence it suffices to show that $x \notin \mathcal{R}(f)$ for $x \in S^2 - (0 \cup 0' \cup S^1)$. Let $\theta = |\theta(x)|$ and $B = \{y \in S^2 \colon |\theta(y)| \leqslant \theta\}$. Then by (*), $|\theta(f(y))| < \theta$ for any $y \in B$. Since B is compact, there exists $\varepsilon > 0$ such that $|\theta(y')| < \theta$ for any $y' \in U_\varepsilon(f(B))$ ($U_\varepsilon(\ldots)$ is an ε-neighborhood of $(\ldots)$). Let $\{x_0, \ldots, x_m\}$ be a sequence of points on S^2 with $x_0 = x$, $d(f(x_i), x_{i+1}) < \varepsilon$. Then $x_1 \in U_\varepsilon(f(B)) \subset B$ since $x_0 = x \in B$. By induction, $x_m \in U_\varepsilon(f(B))$ so that $|\theta(x_m)| < \theta = |\theta(x)|$. Therefore there exists no sequence $\{x_0, \ldots, x_n\}$ of points on S^2 with $x_0 = x_n = x$ and $d(f(x_i), x_{i+1}) < \varepsilon$. Hence $x \notin \mathcal{R}(f)$ and $\mathcal{R}(f) = 0 \cup 0' \cup S^1$.

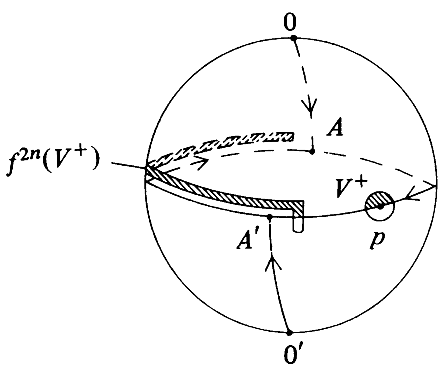

FIGURE 3

We next show that $\mathcal{R}(f) = \Omega(f) \neq \Omega(f^2)$. Clearly $0, 0', A, A' \in \Omega(f)$ $(\in \Omega(f^2))$. Let $p \in S^1 - (A \cup A')$, V be a neighborhood of p and $V^+ = V \cap (H^+ - S^1)$. Without loss of generality, we may assume that $f^{2n}(p) \to A'$ as $n \to \infty$. Then

$f^{2n}(V^+)$ is pressed toward S^1 as in Figure 3. Therefore for a sufficiently large n, $f^{2n+1}(V) \cap V = f^{2n+1}(V^+) \cap V \neq \varnothing$. Hence $p \in \Omega(f)$ and $\mathfrak{R}(f) = \Omega(f)$.

On the other hand, $(f^2)^n(V) \cap V = f^{2n}(V) \cap V = \varnothing$. Hence $p \notin \Omega(f^2)$. Therefore $\mathfrak{R}(f) = \Omega(f) \neq \Omega(f^2)$.

Hence f has no C^0-Ω-explosion but f^2 has Ω-explosion.

References

1. C. Conley, *The gradient structure of flow*. I, IBM research, RC 3932 (#17806), Yorktown Heights, New York, July 17, 1972.

2. M. Shub, *Stabilité globale des systèmes dynamiques*, Asterisque **56** (1976).

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