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ON SEMISIMPLE SEMIGROUP RINGS

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ABSTRACT. Let π be a property of rings that satisfies the conditions that (i) homomorphic images of π-rings are π-rings and (ii) ideals of π-rings are π-rings. Let S be a semilattice P of semigroups S_α. If each semigroup ring $R[S_\alpha]$ ($\alpha \in P$) is π-semisimple, then the semigroup ring $R[S_\alpha]$ is also π-semisimple. Conditions are found on P to insure that each $R[S_\alpha]$ ($\alpha \in P$) is π-semisimple whenever S is a strong semilattice P of semigroups S_α and $R[S]$ is π-semisimple. Examples are given to show that the conditions on P cannot be removed. These results and examples answer several questions raised by J. Weissglass.

Let π be a property of rings. A ring is called a *π-ring* if it has property π. An ideal I of a ring is a *π-ideal* if I is a π-ring. A ring is *π-semisimple* if it has no nonzero π-ideals. Henceforth, we assume that the property π satisfies the conditions, (i) homomorphic images of π-rings are π-rings and (ii) ideals of π-rings are π-rings. For example, the properties of being nil, nilpotent, or left quasi-regular are such properties.

Let R be a ring and let S be a semigroup. Then the *semigroup ring* $R[S]$ consists of all formal sums $\sum_{s\in S} r_s s$ such that $r_s \in R$ and $r_s = 0$ for all but finitely many $s \in S$; addition and multiplication are defined in the obvious manner (see **[1]**, **[2]**, **[3]**, **[6]**, **[7]**). Several authors have studied properties π for specialized semigroup rings. For example, Gilmer and Parker **[2]** compute the nil radical of $R[S]$ under the assumption that $R[S]$ is commutative. Passman **[4]** discusses many radical properties under the assumption that R is a field and S is a group. Krempa **[3]** computes the lower Baer radical of $R[S]$ when S is a cancellative Ω_1-semigroup. Quesada **[6]** also computes the lower Baer radical of $R[S]$ in several interesting cases.

Let P be a semilattice whose natural order is indicated by $>$. As in **[5]**, a semigroup S is called a *strong semilattice* P of semigroups S_α if there exists a family of homomorphisms $\varphi_{\alpha,\beta}$: $S_\alpha \to S_\beta$ ($\alpha, \beta \in P$; $\alpha \geqslant \beta$) satisfying the following conditions.

(a) S is the disjoint union of the semigroups S_α.

(b) $\varphi_{\alpha,\alpha}$ is the identity map S_α.

(c) If $\alpha > \beta > \gamma$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

(d) If $a \in S_\alpha$ and $b \in S_\beta$, then multiplication in S is determined by $a \cdot b = (a\varphi_{\alpha,\alpha\beta}) \cdot (b\varphi_{\beta,\alpha\beta}) \in S_{\alpha\beta}$.

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For more details, see §III.7 of [5]. We note that each $\varphi_{\alpha,\beta}$ has a natural extension to a ring homomorphism from $R[S_\alpha]$ to $R[S_\beta]$:

$$\sum_{s\in S_\alpha} r_s s \to \sum_{s\in S_\alpha} r_s(s\varphi_{\alpha,\beta}).$$

We also denote this extension by $\varphi_{\alpha,\beta}$ for convenience. *For the rest of this paper, S will denote a strong semilattice P of semigroups S_α.*

The purpose of this paper is to study conditions relating the π-semisimplicity of $R[S]$ to the π-semisimplicity of the $R[S_\alpha]$ $(\alpha \in P)$. If each $R[S_\alpha]$ is π-semisimple, then $R[S]$ is π-semisimple. This result is an immediate consequence of Theorem 1, which generalizes [6, Theorem 4.21] and [7, Theorem 1 and Corollary 1] and which answers affirmatively Question 4 of Weissglass [7, p. 477]. However, if $R[S]$ is π-semisimple, not all of the $R[S_\alpha]$ need be π-semisimple. A problem may occur if $R[S_\alpha]$ has a nonzero π-ideal and there are infinitely many $\beta \in P$ closely beneath α in the order $>$ of P. We determine conditions on P in our main result (Theorem 2) which guarantee that $R[S]$ is π-semisimple if and only if $R[S_\alpha]$ is π-semisimple for all $\alpha \in P$. The key part of the proof of Theorem 2 is the construction of a special type of ideal of $R[S]$ from an ideal of $R[S_\alpha]$; this construction is done in Lemma 2. Theorem 2 gives an answer to Question 1 of Wiessglass [7, p. 477] for a semigroup ring case. Examples are given to show that the conditions on P in the hypothesis of Theorem 2 cannot be dropped when π is the property of being nil, nilpotent, or left quasi-regular, even if each S_α is a group and R is a field. Thus these examples answer in the negative Question 7 of Weissglass [7, p. 477]. A modification of these examples is also used to answer in the negative Question 2 of Weissglass [7, p. 477] concerning the Jacobson radical of certain commutative rings.

Many interesting semigroups appear as strong semilattices P of semigroups S_α, where S_α is a special type of semigroup; for example, see [1, Theorem 4.11] and [5, Theorem IV.4.3 and Corollaries IV.4.6, IV.4.7]. Since conditions for many special types of semigroup rings to be π-semisimple are known, our theorems often provide a realistic test for the π-semisimplicity of $R[S]$.

Before stating our first result, we need a generalization of a semigroup ring $R[S']$, where S' is any semilattice of semigroups.

In [7], a ring T is called a *supplementary semilattice sum of subrings* T_α $(\alpha \in P)$ if the following conditions hold: $T = \Sigma_{\alpha\in P} T_\alpha$, $T_\alpha T_\beta \subseteq T_{\alpha\beta}$ for all $\alpha, \beta \in P$, and $T_\alpha \cap (\Sigma_{\alpha\neq\beta} T_\beta) = 0$ for each $\alpha \in P$. Clearly, $R[S]$ is always a supplementary semilattice sum of subrings with $T_\alpha = R[S_\alpha]$. If $t \in T$, we define

$$P\text{-supp } t = \left\{ \alpha \in P \,|\, t = \sum_{\alpha\in P} t_\alpha \text{ and } t_\alpha \neq 0 \right\}.$$

We are now ready to generalize [6, Theorem 4.21] and [7, Theorem 1].

THEOREM 1. *Let $T = \Sigma_{\alpha\in P} T_\alpha$ be a supplementary semilattice sum of subrings. If T_α is π-semisimple for each $\alpha \in P$, then T is also π-semisimple.*

PROOF. Let B be a π-ideal of T. We assume that $0 \neq x \in B$ and seek a contradiction. Clearly there exists a $\beta \in P$ satisfying the following conditions. (a) $\beta \in P$-supp x, and (b) if $\alpha \in P$-supp x, then $\alpha \not> \beta$. (In fact, by [7, Lemma 2] there will be more than one such β.) Let $P' = \{\alpha \in P | \alpha \leq \gamma$ and $\gamma \in P$-supp $x\}$; then P' is a semilattice. Hence $T' = \Sigma_{\alpha \in P'} T_\alpha$ is both a supplementary semilattice sum of subrings and an ideal of T. Let $A = B \cap T'$. Then $x \in A$. Since A is an ideal of the π-ring B, then A is π-ring by our standing hypothesis on π. Since A is also an ideal of T', then A is a π-ideal of T'.

Let θ: $T' \to T_\beta$ be the projection map. Then $0 \neq \theta(x) \in \theta(A)$ by (a). Clearly θ preserves addition. Let $c_1, c_2 \in T'$, and write $c_i = d_i + e_i$ $(i = 1, 2)$, where $d_i \in T_\beta$ and $e_i \in \Sigma_{\alpha \neq \beta} T_\alpha$. By (b), β is maximal in P'; hence $c_1 c_2 = d_1 d_2 + f$, where $\{P\text{-supp} f\} \cap T_\beta = \varnothing$. Thus $\theta(c_1 c_2) = d_1 d_2 = \theta(c_1)\theta(c_2)$. That is, θ is a ring homomorphism of T' onto T_β. But then $\theta(A)$ is a nonzero π-ideal of T_β, which contradicts the hypothesis that T_β is π-semisimple.

We now obtain three results needed in the proof of Theorem 2.

LEMMA 1. *If μ is a zero element of P and if $R[S]$ is π-semisimple, then $R[S_\mu]$ is π-semisimple.*

PROOF. Let A be a π-ideal of $R[S_\mu]$. If $s \in R[S_\alpha]$ for some α and $a \in A$, then $sa = (s\varphi_{\alpha,\mu})a \in A$ and $as = a(s\varphi_{\alpha,\mu}) \in A$. It follows that A is also an ideal of $R[S]$, and hence $A = 0$.

Let α, $\beta \in P$. Then β is said to be *maximal under* α if $\beta < \alpha$ and there is no $\gamma \in P$ such that $\beta < \gamma < \alpha$. Further, P is called an *m.u.-semilattice* if P satisfies the following conditions.

(1) If α, $\delta \in P$ with $\delta < \alpha$, then there exists β maximal under α such that $\delta \leq \beta < \alpha$.

(2) The set $\{\beta \in P | \beta$ is maximal under $\alpha\}$ is finite for each $\alpha \in P$.

LEMMA 2. *Let P be an m.u.-semilattice, and suppose $\alpha \in P$ is not a zero of P. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be the set of elements maximal under α. For each nonempty subset Δ of $\{\beta_1, \beta_2, \ldots, \beta_n\}$, let Δ' denote the product (g.l.b.) of all the β_i in Δ, and let $|\Delta|$ denote the cardinality of Δ. If A is an ideal of $R[S_\alpha]$, then for each $a \in A$, define*

$$a^* = a - \left[\sum_{|\Delta|=1} a\varphi_{\alpha,\Delta'}\right] + \sum_{|\Delta|=2} a\varphi_{\alpha,\Delta'} - \left[\sum_{|\Delta|=3} a\varphi_{\alpha,\Delta'}\right] + \cdots + (-1)^n a\varphi_{\alpha,\beta_1\beta_2\cdots\beta_n}.$$

Then $A^* = \{a^* | a \in A\}$ is an ideal of $R[S]$.

PROOF. Let $a^* \in A^*$, and let $x \in R[S_\gamma]$ for some $\gamma \in P$. We wish to show that $xa^* \in A^*$.

Suppose that $\gamma \geq \alpha$. Since $(x\varphi_{\gamma,\alpha})a \in A$, then

$$xa^* = (x\varphi_{\gamma,\alpha})a - \left[\sum_{|\Delta|=1} ((x\varphi_{\gamma,\alpha})a)\varphi_{\alpha,\Delta'}\right] + \left[\sum_{|\Delta|=2} ((x\varphi_{\gamma,\alpha})a)\varphi_{\alpha,\Delta'}\right] - \cdots + (-1)^n((x\varphi_{\gamma,\alpha})a)\varphi_{\alpha,\beta_1\beta_2\cdots\beta_n} \in A^*.$$

Suppose that $\gamma < \alpha$. Then, after reindexing, $\gamma \leqslant \beta_1 \beta_2 \cdots \beta_k$ for some maximal k, where $1 \leqslant k \leqslant n$. If $\pm a\varphi_{\alpha,\Delta'}$ is a term of a^* and $\Delta \subseteq \{\beta_1, \beta_2, \ldots, \beta_k\}$, then $\pm x(a\varphi_{\alpha,\Delta'}) = \pm x(a\varphi_{\alpha,\gamma})$. The sum of the terms of xa^* having this form is

$$x(a\varphi_{\alpha,\gamma})\left[1 - k + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k\right] = x(a\varphi_{\alpha,\gamma})[1 + (-1)]^k = 0. \quad (\dagger)$$

All other terms of xa^* involve some $\varphi_{\alpha,\Delta'\Gamma'}$, where Δ is a subset of $\{\beta_1, \beta_2, \ldots, \beta_k\}$, Γ is a nonempty subset of $\{\beta_{k+1}, \beta_{k+2}, \ldots, \beta_n\}$, and Γ' is the product (g.l.b.) of the elements of Γ. For a given Γ,

$$x(a\varphi_{\alpha,\Delta'\Gamma'}) = (x\varphi_{\gamma,\gamma\Delta'\Gamma'})(a\varphi_{\alpha,\gamma\Delta'\Gamma'}) = (x\varphi_{\gamma,\gamma\Gamma'})(a\varphi_{\alpha,\gamma\Gamma'}) = x(a\varphi_{\alpha,\gamma\Gamma'})$$

for all Δ. In xa^*, the sign of the $\Delta'\Gamma'$ term is equal (opposite) that of the Γ' term if $|\Delta|$ is even (odd). Hence, for a given Γ, the sum over Δ of the $\Delta'\Gamma'$ terms is

$$x(a\varphi_{\alpha,\gamma\Gamma'})\left[1 - k + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k\right] = 0. \quad (\dagger\dagger)$$

Since every term of xa^* appears exactly once in an expression ($\dagger$) or ($\dagger\dagger$), then $xa^* = 0 \in A^*$.

Suppose that α and γ are not related by $<$; i.e., suppose that neither $\alpha \leqslant \gamma$ nor $\alpha > \gamma$ holds. Then for all nonempty $\Delta \subseteq \{\beta_1, \beta_2, \ldots, \beta_n\}$,

$$x(a\varphi_{\alpha,\Delta'}) = (x\varphi_{\gamma,\gamma\Delta'})(a\varphi_{\alpha,\gamma\Delta'}) = (x\varphi_{\gamma,\alpha\gamma}\varphi_{\alpha\gamma,\alpha\gamma\Delta'})(a\varphi_{\alpha,\alpha\gamma\Delta'}) = (x\varphi_{\gamma,\alpha\gamma})(a\varphi_{\alpha,\Delta'}).$$

If Δ is the empty set, define $x(a\varphi_{\alpha,\Delta'}) = xa$. Thus we may replace x by $x\varphi_{\gamma,\alpha\gamma}$ in this case; hence it follows from the preceding paragraph that $xa^* = 0 \in A^*$.

The three preceding paragraphs show that A^* is closed under left multiplication by elements of any S_γ and hence by elements of $R[S]$. A symmetric argument shows that A^* is closed under right multiplication by elements of $R[S]$. Clearly A^* is closed under addition. Thus A^* is an ideal of $R[S]$.

LEMMA 3. *Under the hypotheses of Lemma 2, the mapping* $*: A \to A^*: a \to a^*$ *is a ring isomorphism of* A *onto* A^*.

PROOF. Let $a_1, a_2 \in A$. Then $a_1{}^*a_2{}^* = a_1a_2{}^*$ + terms of the form $(a_1\varphi_{\alpha,\gamma})a_2{}^*$, where $\gamma < \alpha$. It follows from the case $\gamma \geqslant \alpha$ in the proof of Lemma 2 that $a_1a_2{}^* = (a_1a_2)^*$. It follows from the case $\gamma < \alpha$ in the proof of Lemma 2 that each of the remaining terms $(a_1\varphi_{\alpha,\gamma})a_2{}^*$ is 0. Hence $a_1{}^*a_2{}^* = a_1a_2{}^* = (a_1a_2)^*$. The formula $a_1{}^* + a_2{}^* = (a_1 + a_2)^*$ follows from the commutativity of addition and the homomorphism properties of the $\varphi_{\alpha,\beta}$. Clearly, $*$ is a 1-1 map.

We are now ready for our main result.

THEOREM 2. *Let* P *be an m.u.-semilattice. Then* $R[S]$ *is* π*-semisimple if and only if* $R[S_\alpha]$ *is* π*-semisimple for each* $\alpha \in P$.

PROOF. The "if" part follows from Theorem 1.

For the "only if" part, let A be a π-ideal of $R[S_\alpha]$ for some α. If α is a zero of P, then $A = 0$ by Lemma 1. If α is not a zero of P, then $A^* = \{a^* | a \in A\}$ is a π-ideal of $R[S]$ by Lemma 2 and 3; hence we must also have $A = 0$ in this case. Therefore, $R[S_\alpha]$ is π-semisimple.

Question 1 [7, p. 477] is as follows: Find a condition on P such that if T is a supplementary semilattice sum of T_α ($\alpha \in P$), then T π-semisimple implies T_α is π-semisimple for each $\alpha \in P$. Theorem 2 states that P being an m.u.-semilattice is such a condition when $T = R[S]$.

COROLLARY 1. *Let P be an m.u.-semilattice. Then $R[S]$ is semiprime if and only if $R[S_\alpha]$ is semiprime for each $\alpha \in P$.*

COROLLARY 2. *Let P be an m.u.-semilattice. Then $R[S]$ is nil semisimple if and only if $R[S_\alpha]$ is nil semisimple for each $\alpha \in P$.*

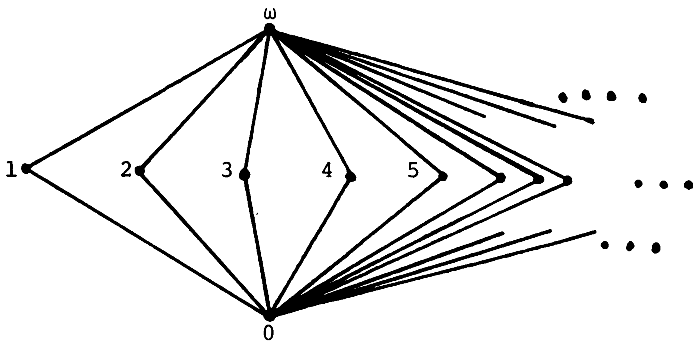
COROLLARY 3. *Let P be an m.u.-semilattice. Then $R[S]$ is Jacobson semisimple if and only if $R[S_\alpha]$ is Jacobson semisimple for each $\alpha \in P$.*

The condition that P be an m.u.-semilattice cannot be dropped from the hypothesis of Theorem 2. In fact, it is possible for $R[S]$ to be Jacobson semisimple and for some $R[S_\alpha]$ to have a nonzero nilpotent ideal if either of the two conditions in the definition of m.u.-semilattice is dropped. Example 1 (Example 2) shows that condition (1) (condition (2)) in the definition of m.u.-semilattice cannot be dropped. Both examples answer in the negative Question 7 of Weissglass [7, p. 477].

The *support* of $x = \Sigma_{s \in S} r_s s \in R[S]$, denoted by supp x, is the set $\{s \in S | r_s \neq 0\}$.

EXAMPLE 1. Let R be a field of characteristic $p > 0$, let Z_p be the cyclic group of p elements, and let C denote the complex numbers. Let $P = [0, \omega]$, the initial ordinal segment terminating at the first infinite ordinal ω. For $0 \leq n < \omega$, let $S_n = Z_p \wr C$ (wreath product); define S_ω to be a direct sum of C copies of Z_p. We use the following maps $\varphi_{\alpha,\beta}$ to define $S = \bigcup_{\alpha \leq \omega} S_\alpha S_\alpha$ as a semilattice P of semigroups: $\varphi_{\alpha,\beta}$ is the identity map on $Z_p \wr C$ for $0 \leq \beta \leq \alpha < \omega$, and $\varphi_{\omega,\beta}$ is the natural embedding of S_ω as a normal subgroup of $S_\beta = Z_p \wr C$ for $0 \leq \beta < \omega$. From [4, Lemma VII.4.12] it follows that $R[S_\alpha]$ is Jacobson semisimple for $0 \leq \alpha < \omega$. Since S_ω is an Abelian p-group, then $R[S_\omega]$ has a nonzero nilpotent ideal by [4, Theorem IV.2.13]. Let $K = \Sigma_{\alpha < \omega} R[S_\alpha]$. Then K is an ideal of $R[S]$; moreover, K is a Jacobson semisimple ring by Theorem 1. If N is a quasi-regular ideal of $R[S]$, then $N \cap K = 0$. Thus, if $0 \neq x \in N$, then $\{\text{supp } x\} \cap S_\omega \neq \varnothing$. If $\{\text{supp } x\} \cap S_\alpha \neq \varnothing$ for some $\alpha < \omega$, let $m = 1 + \max\{\alpha | 0 \leq \alpha < \omega$ and $\{\text{supp } x\} \cap S_\alpha \neq \varnothing\}$; otherwise, let $m = 1$. Let e_m denote the identity element of S_m. Then $\{\text{supp } e_m x\} \cap S_m \neq \varnothing$ and $e_m x \in K \cap N$, which contradicts the fact that $K \cap N = 0$. Hence $R[S]$ is a Jacobson semisimple ring.

EXAMPLE 2. Let $P = [0, \omega]$ as a *set*, but impose a new order $\gg$ on P (rather than the usual order $>$) by the following rules: (i) if m, n, ω are distinct elements of P, then $m \wedge n = 0$ in the order $\gg$; (ii) if $m \neq \omega$, then $\omega \gg m$. Thus P can be made into a semilattice with the following order structure.



Now let R, S_α, $\varphi_{\alpha,\beta}$, S, K, and N be defined as in Example 1, except that $\gg$ replaces $>$ in the definition of these terms. Thus $R[S_\alpha]$ is Jacobson semisimple whenever $\omega \gg \alpha$, $R[S_\omega]$ has a nonzero nilpotent ideal, and $K \cap N = 0$ for any quasi-regular ideal N of $R[S]$. Thus if $0 \neq x \in N$, then $\{\text{supp } x\} \cap S_\omega \neq \varnothing$. Since $\{\text{supp } x\}$ is finite, there exists $m \in P - \{0, \omega\}$ such that $\{\text{supp } x\} \cap S_m = \varnothing$. Let e_m denote the identity element of S_m. Then $\{\text{supp } e_m x\} \cap S_m \neq \varnothing$ and $e_m x \in K \cap N$, which contradicts the fact that $K \cap N = 0$. Hence $R[S]$ is a Jacobson semisimple.

J. Weissglass has asked the following question [7, Question 2, p. 477]. If $T = \Sigma_{\alpha \in \Omega} T_\alpha$ is a supplementary semilattice sum and T is commutative, does T Jacobson semisimple imply that each T_α is Jacobson semisimple? We apply our methods to answer this question in the negative in the following example.

Example 3. Let P be the semilattice of Example 1. Let T_ω be a commutative local integral domain with identity element that is not a field. Let T_n be the quotient field of T_ω for $0 \leqslant n < \omega$. Define the homomorphisms $\varphi_{\alpha,\beta}$ to be the identity map on T_n for $0 \leqslant \beta \leqslant \alpha < \omega$; define $\varphi_{\omega,\beta}$ to be the natural embedding of T_ω into T_β for each $\beta \leqslant \omega$. Then $T = \Sigma_{\beta \leqslant \omega} T_\beta$ forms a commutative supplementary semilattice sum of subrings T_β by naturally extending the multiplication $x \cdot y = \varphi_{\alpha,\alpha\wedge\beta}(x) \cdot \varphi_{\beta,\alpha\wedge\beta}(y)$ for $x \in T_\alpha$, $y \in T_\beta$. Now T_ω is not Jacobson semisimple. Since T_β is Jacobson semisimple for each $\beta < \omega$, then $K = \Sigma_{\beta<\omega} T_\beta$ is Jacobson semisimple by Theorem 1. If N is a quasi-regular ideal of T, then $K \cap N = 0$. If $0 \neq x \in N$, then $\omega \in P$-supp x. If $\alpha \in \{P\text{-supp } x\} \neq \varnothing$ for some $\alpha < \omega$, let $m = 1 + \max\{\alpha | 0 \leqslant \alpha < \omega \text{ and } \alpha \in \{P\text{-supp } x\}\}$; otherwise, let $m = 1$. Let e_m denote the identity element of T_m. Then $m \in P$-supp $e_m x$ and $e_m x \in K \cap N$, which contradicts the fact that $K \cap N = 0$. Hence T is a Jacobson semisimple ring.

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