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NO SYSTEM OF UNCOUNTABLE RANK IS PURELY SIMPLE

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ABSTRACT. A pair of complex vector spaces (V, W) is a system if and only if there is a C-bilinear map $\mathbf{C}^2 \times V$ to W. The category of systems is equivalent to the category of modules over a certain subring of the ring of 3×3 matrices over the complex numbers, and so module-theoretic concepts make sense for systems. A system is purely simple if it has no proper pure subsystem. Recently it has been shown that for every positive integer n, there exists a purely simple system of rank n but no system of rank greater than the cardinality of the continuum is purely simple. In this paper it is shown that no system of rank greater than $\aleph_0$ is purely simple. Necessary and sufficient conditions for a system of rank $\aleph_0$ to be purely simple are also given.

We shall assume familiarity with the notations and terminology in [3].

LEMMA. *Let (V, W) be a nonzero torsion-free system containing no nonzero pure subsystem of finite rank. Then (V, W) contains an ascending sequence of subsystems*

$$(V_1, W_1) \subset (V_2, W_2) \subset \cdots \subset (V_m, W_n) \subset \cdots$$

such that

(1) *each (V_n, W_n) is torsion-closed in (V, W) and is a finite direct sum $(V_n, W_n) = \sum_{j=1}^{r_n}(V_{nj}, W_{nj})$ with (V_{nj}, W_{nj}) of type* $\mathrm{III}^{m_{nj}}$ *and* $(\min\{m_{nj}\colon 1 \leqslant j \leqslant r_n\})_{n=1}^{\infty}$ *unbounded.*

PROOF. Since (V, W) is nonzero and torsion-free, there exists a nonzero w in W. Put $(V_1, W_1) = \mathrm{tc}_{(V,W)}(\varnothing, \{w_1\})$. By Theorem 1 of [2], if (V_1, W_1) were infinite-dimensional, it would be a nonzero pure subsystem of (V, W) of rank 1, contradicting the assumption on (V, W). Hence (V_1, W_1) is a torsion-closed subsystem of type $\mathrm{III}^{m_{11}}$, with $m_{11} \geqslant 1$.

For $n \geqslant 1$ suppose that there exists a sequence of finite-dimensional torsion-closed subsystems $(V_1, W_1) \subset \cdots \subset (V_n, W_n)$, where, with the decompositions as in (1),

$$\min\{m_{kj}\colon 1 \leqslant j \leqslant r_k\} \geqslant k \quad \text{for } k = 1, \ldots, n.$$

By the assumption on (V, W), it has no direct summand of type III^m for any positive integer m. Therefore, by Theorem 2 of [2], (V_n, W_n) is contained in a subsystem (U, Z) of (V, W) which is a finite direct sum $(U, Z) = \sum_{j=1}^{s} \cdot (U_j, Z_j)$ with (U_j, Z_j) of type III^{p_j} and $\min\{p_j\colon 1 \leqslant j \leqslant s\} \geqslant n + 1$. Put $(V_{n+1}, W_{n+1}) = \mathrm{tc}_{(V,W)}(U, Z)$. We have $\mathrm{rank}(U, Z) = s$. If $\{z_j\}_{j=1}^{s}$ is a subset of Z such that

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$$(U, Z) = \mathrm{tc}_{(U,Z)}(\varnothing, \{z_j\}_{j=1}^{s}),$$

then

$$(V_{n+1}, W_{n+1}) = \mathrm{tc}_{(V,W)}(\varnothing, \{z_j\}_{j=1}^{s}).$$

Therefore if $r_{n+1} = \operatorname{rank}(V_{n+1}, W_{n+1})$, then $r_{n+1} \leqslant s$. Since by assumption (V_{n+1}, W_{n+1}) is not pure in (V, W), it follows from [2, Theorem 1] that (V_{n+1}, W_{n+1}) has a direct summand $(V_{n+1,1}, W_{n+1,1})$ of some type $\mathrm{III}^{m_{n+1,1}}$. Since the rank of a direct complement of $(V_{n+1,1}, W_{n+1,1})$ in (V_{n+1}, W_{n+1}) is $r_{n+1} - 1$, it follows by induction on r_{n+1} that (V_{n+1}, W_{n+1}) is a direct sum,

$$(V_{n+1}, W_{n+1}) = \sum_{j=1}^{r_{n+1}} (V_{n+1,j}, W_{n+1,j})$$

with $(V_{n+1,j}, W_{n+1,j})$ of type $\mathrm{III}^{m_{n+1,j}}$.

To complete the inductive existence proof of the required sequence of subsystems, it suffices to show that $\min\{m_{n+1,j}\colon 1 \leqslant j \leqslant r_{n+1}\} \geqslant \min\{p_j\colon 1 \leqslant j \leqslant s\}$. Suppose to the contrary that (V_{n+1}, W_{n+1}) has a direct decomposition

$$(V_{n+1}, W_{n+1}) = (X_1, Y_1) \dotplus (X_2, Y_2)$$

with (X_1, Y_1) of type III^m and $m < p_j$ for all $j = 1, \ldots, s$. Let (π, ρ) be the projection of (V_{n+1}, W_{n+1}) onto (X_1, Y_1) with kernel (X_2, Y_2), and for $j = 1, \ldots, s$, let Γ_j be a chain of type III^{p_j} spanning (U_j, Z_j). Then by [1, Theorem 6.3(c)], $(\pi, \rho)\Gamma_j = 0$. Hence each (U_j, Z_j) and therefore also (U, Z) is contained in (X_2, Y_2). Since (X_2, Y_2) is torsion-closed in (V_{n+1}, W_{n+1}), which in turn is torsion-closed in (V, W), (X_2, Y_2) is torsion-closed in (V, W). Hence $(V_{n+1}, W_{n+1}) = \mathrm{tc}_{(V,W)}(U, Z) \subset (X_2, Y_2)$, a contradiction.

THEOREM 1. *Every nonzero system (V, W) has a nonzero pure subsystem of rank $\leqslant \aleph_0$. Hence a system of rank $> \aleph_0$ cannot be purely simple.*

PROOF. By [1, Proposition 9.12], $t(V, W)$ is pure in (V, W). So if $t(V, W) \neq 0$, (V, W) has a nonzero pure subsystem of rank 0. Thus we may assume that (V, W) is torsion-free. If (V, W) has a nonzero pure subsystem of finite rank, we are done. Otherwise, put $(X, Y) = \bigcup_{n=1}^{\infty}(V_n, W_n)$, where the sequence $((V_n, W_n))_{n=1}^{\infty}$ is as described in the Lemma. As a union of an ascending sequence of torsion-closed subsystems of (V, W), (X, Y) is torsion-closed in (V, W). Suppose that (X, Y) has a direct summand (U, Z) of type III^m. Then for all n beyond some n_0, (U, Z) is contained in (V_n, W_n), and hence is a direct summand of (V_n, W_n). Since any two decompositions of (V_n, W_n) into a direct sum of indecomposable subsystems are isomorphic (see [1, p. 309]), this would imply that $\min\{m_{nj}\colon 1 \leqslant j \leqslant r_n\} \leqslant m$ for all $n > n_0$, contradicting the unboundedness condition. Hence (X, Y) contains no such (U, Z), and by [2, Theorem 1], (X, Y) is pure in (V, W). By construction $\bigcup_{n=1}^{\infty} W_n$ has a linear countable basis $\{w_k\}$. Since $(X, Y) = \mathrm{tc}_{(X,Y)}(\varnothing, \{w_k\})$, a subset L of this basis is a basis of (X, Y) with respect to generation. Since (V, W) has no nonzero pure subsystem of finite rank, L is infinite and rank $(X, Y) = \aleph_0$.

THEOREM 2. *A torsion-free system is purely simple of rank* $\aleph_0$ *if and only if*

(a) (V, W) *is the union of an ascending sequence of subsystems* $(V_1, W_1) \subset (V_2, W_2) \subset \cdots$ *satisfying* (1) *and*

(b) *every nonzero proper torsion-closed subsystem of* (V, W) *has a direct summand of type* III^m *for some positive integer* m.

PROOF. If (V, W) is purely simple of rank $\aleph_0$, it has no nonzero pure subsystem of finite rank. Hence, as in the proof of Theorem 1, (V, W) contains a nonzero pure subsystem $(X, Y) = \bigcup_{n=1}^{\infty}(V_n, W_n)$ where $((V_n, W_n))_{n=1}^{\infty}$ is an ascending sequence satisfying (1). Since (V, W) is purely simple, $(X, Y) = (V, W)$ and (a) holds. By [**2**, Theorem 1], a nonzero torsion-closed subsystem with no direct summand of type III^m is pure in (V, W), and hence cannot be proper. This establishes (b).

Conversely, suppose that (V, W) satisfies (a) and (b) and has a nonzero pure subsystem (S, T). (S, T) is then torsion-closed in (V, W), and so by (b) has a direct summand (U, Z) of type III^m. By transitivity of purity (U, Z) is pure in (V, W), hence by [**1**, Theorem 5.5] it is a direct summand of (V, W). Since (V, W) has the same kind of structure as (X, Y) in the proof of Theorem 1, such a direct summand (U, Z) is impossible. So (V, W) must be purely simple. As in the proof of Theorem 1, $\mathrm{rank}(V, W) \leqslant \aleph_0$. If (V, W) were of finite rank, it would be of the form $(V, W) = \mathrm{tc}_{(V,W)}(\varnothing, L)$ where L is a finite subset of W. Then $L \subset W_{n_0}$ for some n_0, and since (V_{n_0}, W_{n_0}) is torsion-closed, $(V, W) \subset (V_{n_0}, W_{n_0})$. However, by the unboundedness condition the sequence of (a) cannot be stationary. This contradiction shows that $\mathrm{rank}(V, W) = \aleph_0$.

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