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EXTENSIONS OF DIFFERENCE SPECIALIZATIONS

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ABSTRACT. Maximal difference specializations and difference places are defined. Let R be the domain of a difference specialization ϕ of a difference field K and $x \in K$. Then ϕ can be extended to a specialization $x \to 0$ if and only if $1 \notin [x]$. This result applies to give a condition on a polynomial for the extension of a specialization to its generic zero. In a slightly different direction, a necessary and sufficient condition for the extension of a specialization to a larger difference field is given.

Introduction. Numerous examples (see [3] and below) have shown that an extension of a difference specialization may be impossible while the corresponding algebraic extension is easy to obtain. When difference extensions exist remains a question of continued interest.

This work provides conditions for the extension of a specialization to one sending a particular element to 0 (Theorems 1 and 2). It is then possible to provide, in difference algebra, conditions for extensions of specializations to extension fields (Theorems 5 and 6), which are analogous to conditions given by S. D. Morrison in differential algebra [5].

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The basic definitions of difference algebra are assumed [3]. In a difference ring R, the difference ideal generated by a set N is denoted by $[N]$, while the perfect difference ideal is denoted $\{N\}$. Let $N^0 = N$, let $N' = \{a \in R |$ some power product of transforms of a is in $N\}$ and let $N^{(k+1)} = [N^{(k)}]'$, $k = 1, 2, \ldots$. Then $\{N\} = \bigcup_{k=0}^{\infty} N^{(k)}$. If R is an integral domain, then any difference homomorphism of R into a difference field is called a difference specialization of R.

1. Difference places. Let K be a difference field with transforming operator τ. A *maximal difference specialization* of K is a difference homomorphism ϕ of a difference subring of K onto a difference domain Λ, which cannot be extended to a difference homomorphism of a larger difference subring of K onto a difference domain extension of Λ. It may be noted that Λ is, in fact, a field. If $\phi(x) \neq 0$, then $\phi(x^{-1})$ can be defined by $\phi(x)^{-1}$ in the field of quotients of Λ, but ϕ is maximal, so $\phi(x)^{-1} \in \Lambda$. The domain R of ϕ is called a *maximal difference ring* of K. If K is the quotient field of R, then R is called a *difference valuation ring* of K and ϕ is called a *difference place* of K. If K is an inversive difference field, then any difference homomorphism of a subring of K can be extended to its inversive closure in K, so any maximal difference ring of an inversive field is inversive.

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Let ϕ be a maximal difference specialization with domain R and let $M(R)$ denote the kernel of ϕ. $M(R)$ is a prime reflexive difference ideal, which consists of the nonunits of R. Thus R is a local ring with maximal ideal $M(R)$. More generally, a *local difference ring* is defined to be a difference ring whose nonunits form a difference ideal. Given any difference ring R with prime difference ideal P, the local ring $R_P = \{r/s | s \notin P\}$ is a local difference ring if and only if P is reflexive. This follows from the fact that P is reflexive if and only if $s \notin P$ implies $s_1 \notin P$. Consequently, the maximal ideal $M(R)$ of a local difference ring is reflexive, since in this case $R = R_{M(R)}$.

For any local difference subring of a difference field K, the following are equivalent: (i) R is a maximal difference ring of K; (ii) R is maximal among local difference subrings of K, ordered by domination; (iii) if $x \in K$ and $x \notin R$, then $1 \in \{R\{x\}M(R)\}$, the perfect difference ideal generated by $M(R)$ in $R\{x\}$. The equivalence of (iii) follows from the fact that every proper perfect difference ideal is contained in a prime reflexive difference ideal.

Let R a difference valuation ring of K. Then the set U of units of R forms a subgroup of $K^* = K - \{0\}$ and the natural homomorphism v: $K^* \to K^*/U$ may be defined. Let K^*/U be denoted by Γ, with the operation written as addition. Then v will be called a *difference valuation* of K. Let $\Gamma^+ = v(M(R)^*)$; then for $a \in \Gamma^+$, $-a \notin \Gamma^+$. For a and $b \in \Gamma$, define $a < b$ if $b - a \in \Gamma^+$. Then Γ is an ordered group, but is not necessarily totally ordered. $x \in R^*$ if and only if $v(x) \geqslant 0$ and $x \in M(R)^*$ if and only if $v(x) > 0$.

The following example, due to R. M. Cohn, shows that there are difference valuation rings which are not valuation rings (and, thus, difference valuations which are not valuations). Let $\mathbf{C}$ be the set of complex numbers and let a be transcendental over $\mathbf{C}$. Consider $\mathbf{C}(a) = \mathbf{C}\langle a \rangle$ as a difference field by defining $a_1 = -a$. Let $\mathbf{C}\langle a \rangle\{y\}$ be a difference polynomial ring and denote the transform of y by y_1. Let $P = y^2 - y_1^2 + ay^2y_1^2$. Then $P + P_1 = (y + y_2)(y - y_2)(1 + ay_1^2)$ and the variety $\mathfrak{M}(P)$ has 2 components, one satisfying $y + y_2$ and the other satisfying $y - y_2$. Let x be a generic zero of an irreducible component, with $x_2 = -x$. For any difference polynomial A, x satisfies A if and only if x satisfies a first-order difference polynomial B (obtained by substituting $-y$ for y_2), which is a multiple of P. It follows that x specializes to 0 over $\mathbf{C}\langle a \rangle$. This specialization can be extended to a maximal one and, hence, there is a difference valuation ring R of $\mathbf{C}\langle a, x \rangle$, with $x \in M(R)$. However, R is not a valuation ring: $z = x_1/x$ is integral over R since z satisfies $1 - Z^2 + ax_1^2 \in R[Z]$, but $z \notin R$. If ϕ is the maximal specialization and $\phi(z)$ is defined, then $\phi(z^2) = \phi(1 + ax_1^2) = 1$ and $\phi(zz_1) = 1$. But $zz_1 = x_1/x \cdot x_2/x_1 = -1$. Thus, z is not in the domain of ϕ.

2. Extensions of a specialization to an element. The following criterion provides a tool in specialization problems, as well as in the development of the theory of difference places.

THEOREM 1. *Let R be a local difference subring of a difference field K and let $x \in K$. The homomorphism ϕ: $R \to R/M(R)$ extends to one sending x to 0 if and only if $1 \notin [x]$ in $R\{x\}$.*

PROOF. If ϕ extends to a homomorphism ϕ' of $R\{x\}$ with $\phi'(x) = 0$, then $[x] \subset \operatorname{Ker} \phi'$. Therefore $1 \notin [x]$.

For the converse, let N denote the difference ideal generated by $M(R)$ and x in $R\{x\}$. Then $N = R\{x\}M(R) + [x] = M(R) + [x]$. If ϕ cannot be extended to a homomorphism of $R\{x\}$ sending x to 0, then $1 \in \{N\}$. It will be shown by an inductive argument that if there is $c \in \{N\}$ with $c = u + z$, $u \in U = R - M(R)$, $z \in [x]$, then $1 \in [x]$. Since $1 \in \{N\}$ and 1 is of this form (with $z = 0$), the proof will be complete.

$\{N\} = \bigcup_{k=0}^{\infty} N^{(k)}$, so $c \in [N^{(k)}]$ for some k. If $k = 0$, then $c \in N = M(R) + [x]$. Thus $c = u + z = m + w$, where $m \in M(R)$ and $w \in [x]$. Since $u \notin M(R)$, $u - m \notin M(R)$ and $(u - m)^{-1} \in R$. Then $1 = (u - m)^{-1}(u - m) = (u - m)^{-1}(w - z) \in [x]$. If $k = n + 1$, then $c = \sum_{i=1}^{p} f_i q_i$ where $f_i \in R\{x\}$ and $q_i \in N^{(n+1)}$. For each i, $f_i = r_i + z_i$ and $q_i = s_i + y_i$, where r_i and s_i are in R and z_i and y_i are in $[x]$. So $\sum_{i=1}^{p} f_i q_i = \sum_{i=1}^{p} (r_i + z_i)(s_i + y_i) = t + w$, where $t = \sum_{i=1}^{p} r_i s_i$ and $w \in [x]$. Thus, $c = u + z = t + w$. If $t \in M(R)$, then, as above, $(u - t)^{-1} \in R$ and $1 \in [x]$. Otherwise, $s_i \notin M(R)$ for some i. For this i, denote $q_i = s_i + y_i$ simply by $q = s + y$. Since $q \in N^{(n+1)}$, there is a product $\pi(q) = (s + y)^{p_0}(\tau s + \tau y)^{p_1} \cdots (\tau^j s + \tau^j y)^{p_j} \in [N^{(n)}]$. But $\pi(q) = \bar{u} + \bar{z}$, where $\bar{u} = s^{p_0}(\tau s)^{p_1} \cdots (\tau^j s)^{p_j}$ and $\bar{z} \in [x]$. Since $s \notin M(R)$ and $M(R)$ is prime and reflexive, $\bar{u} \in R - M(R)$. Therefore, by induction on n, $1 \in [x]$.

COROLLARY. *Let R be a maximal difference ring of K, and let $x \in K$. Then $x \in M(R)$ if and only if $1 \notin [x]$.*

Let $K\{y\}$ be a difference polynomial ring with transform τ and let ϕ be a difference specialization of K with domain R. Let $g(y) \in K\{y\}$ and let x be a generic zero of a component of $\mathfrak{M}(g)$. The specialization of the coefficients of g to 0 does not guarantee that ϕ can be extended to a specialization of x to 0. R. M. Cohn has noted that even if $g(y)$ is of the form $y\tau y + b$, this may not be possible: let b be a nonzero solution of the polynomial Q on p. 332 of [**3**]. However, a sufficient condition for such an extension can be given.

Let R be a difference subring of a difference field K. Let $g(y)$ be a difference polynomial in $R\{y\}$. Let $\{g\}_R$ and $\{g\}_K$ be the perfect difference ideals generated by g in $R\{y\}$ and $K\{y\}$, respectively. The term of the polynomial which has no power of y or its transforms is called the constant term.

THEOREM 2. *Let $g(y) \in R\{y\}$ have constant term $b \in R$. Let $\{g\}_K$ be prime and let x be a generic zero of $\{g\}_K$. Let ϕ: $R \to \Lambda$ be a difference specialization of K with $\phi(b) = 0$. ϕ can be extended to $R\{x\}$ with $\phi(x) = 0$ if $\{g\}_K \cap R\{y\} = \{g\}_R$.*

PROOF. It can be shown by induction that if $f(y) \in \{g\}_R$ and has constant term c, then $\phi(c) = 0$. $\{g\}_R = \bigcup_{k=0}^{\infty}[g]^{(k)}$, so $f \in [g]^{(k)}$ for some k. If $k = 0$, $f(y) = \sum_{i=1}^{p} f_i(y)\tau^i g(y)$, $f_i \in R\{y\}$. Comparison of constant terms yields $c = \sum_1^p r_i \tau^i b$, $r_i \in R$, and hence $\phi(c) = 0$. If $k = n + 1$, then $f(y) = \sum_1^p f_i(y)q_i(y)$, where $f_i \in R\{y\}$ and a power product $\pi_i(q_i) \in [g]^{(n)}$. If q_i has constant term a_i, then $\pi_i(q_i)$ has constant term $\pi_i(a_i)$ and by induction, $\phi(\pi_i(a_i)) = 0$. Hence $\phi(a_i) = 0$ and $\phi(c) = \phi(\sum r_i a_i) = 0$. Consequently, if $f(y) \in \{g\}_R$, its constant term $\neq 1$.

By extending ϕ in K, one may assume that R is a local difference subring of K and hence of $K\{x\}$, with $\{g\}_K \cap R\{y\} = \{g\}_R$. (For, if $P = \text{Ker}\,\phi$ and R is replaced by $S = R_P$, it follows that $\{g\}_K \cap S\{y\} = \{g\}_S$.) By Theorem 1, if ϕ does not extend to $x \to 0$, then $1 \in [x]$ in $R\{x\}$, i.e. $1 = \Sigma b_i(x)\tau^i(x)$, $b_i \in R\{x\}$. Then $f(y) = 1 - \Sigma b_i(y)\tau^i(y) \in \{g\}_K \cap R\{y\}$. Since f has constant term 1, $f \notin \{g\}_R$. Hence, $\{g\}_K \cap R\{y\} \neq \{g\}_R$.

The corollary to Theorem 1 applies to maximal difference rings, yielding the following development.

PROPOSITION 1. *Let R and S be maximal difference rings of K with $R \subset S$. Then $M(S) \subset M(R)$.*

PROOF. If $x \in M(S)$, then $1 \notin [x] \cdot S\{x\}$. But, since $R \subset S$, $1 \notin [x] \cdot R\{x\}$. Hence $x \in M(R)$.

It follows that if R and S are maximal difference rings of K with $R \subset S$, then every ideal of S is an ideal of R.

PROPOSITION 2. *Let R be a maximal difference ring of K, and let P be a prime reflexive difference ideal of R. Then there is a maximal difference ring S of K such that $R \subset S$ and $M(S) = P$.*

PROOF. Let S be a maximal local difference ring of K, dominating R_P. By Proposition 1, $M(S) \subset M(R)$. Hence $M(S) = M(S) \cap R = (M(S) \cap R_P) \cap R = P \cdot R_P \cap R = P$.

The next proposition follows from this with a proof similar to that given in the differential case in [5, Proposition 3].

PROPOSITION 3. *Let R be a maximal difference ring of K. Then the prime reflexive difference ideals of R are linearly ordered by inclusion.*

As a consequence, in a maximal difference ring, every perfect difference ideal is prime. The *difference rank* of a difference valuation ring is defined to be the number of prime reflexive ideals in the ring.

PROPOSITION 4. *Let S be a maximal difference ring of K with specialization ϕ: $S \to \Lambda$. If R is a maximal difference ring of K with $R \subset S$, then $\phi(R)$ is a maximal difference ring of Λ.*

PROOF. Let ψ: $R \to \Omega$ be a maximal specialization of K with domain R. Since $\text{Ker}\,\phi = M(S) \subset M(R) = \text{Ker}\,\psi$, θ: $\phi(R) \to \Omega$ may be defined by $\theta(\phi(r)) = \psi(r)$, with $\text{Ker}\,\theta = \phi(M(R))$. Let $x \in \Lambda$; if $x \notin \phi(R)$, there is $s \in S$, $s \notin R$, such that $\phi(s) = x$. Since ψ is maximal, $1 \in \{R\{s\} \cdot M(R)\}$ and thus, $1 \in \{\phi(R)\{x\} \cdot \text{Ker}\,\theta\}$. Therefore θ cannot be extended to x.

3. Extensions of a place to an extension field. If ϕ is a difference place of K and L is a difference field extension of K, there arises the question of whether ϕ can be extended to a difference place of L. A necessary and sufficient condition is provided in the corollary below. The development is analogous to that of the differential case [5].

THEOREM 3. *Let R_0 be a difference subring of K with prime reflexive ideals P and Q, $P \subset Q$ and let S be a proper, maximal difference ring of K with $R_0 \subset S$ and $M(S) \cap R_0 = P$. Then there is a proper maximal difference ring R with $R_0 \subset R$ and $M(R) \cap R_0 = Q$. Furthermore, if S is a difference valuation ring of K then R is also.*

PROOF. Similar to Theorem 1 [5].

COROLLARY 1. *Let R_0 be a local difference subring of a difference field K. Let L be a difference field extension of K and let S be proper maximal difference ring (valuation ring) of L containing R_0. Then there is a proper maximal difference ring (valuation ring) R of L dominating R_0.*

The existence of S in the corollary is equivalent to the condition that L have a subring S_0, $R_0 \subset S_0$, which contains a proper nonzero prime reflexive difference ideal. That the condition does not always hold is easily seen. For example, if $\mathbf{Q}\{b\}$ is a difference ring with b transcendental over the rationals $\mathbf{Q}$ and $\tau b = b$, the difference specialization $b \to 1$ does not extend to a difference place of $\mathbf{Q}\langle a\rangle$, where $a^2 = b$ and $\tau a = -a$. (See [3, Chapter 7, Example 3].) However, the condition does hold in the situations of Theorems 5 and 6. Theorem 4 is needed for these results.

THEOREM 4. *Let R be a difference integral domain with quotient field K. If $K\langle a_1, \ldots, a_n\rangle$ is a difference field extension of K which is a primary extension, then there is $u \in R$, $u \neq 0$, such that any specialization ϕ of R with $\phi u \neq 0$ can be extended to $R\{a_1, \ldots, a_n\}$.*

This theorem is a slight modification of part of Theorem IV of [3, Chapter 7] (where $R = F\{b_1, \ldots, b_m\}$, F is a difference field and the specializations are over F). The generalization to an arbitrary domain R can be obtained by using Proposition 9, Chapter 0 of [4] in the lemmas preceding the theorem.

THEOREM 5. *Let R_0 be a local difference ring with quotient field K and no minimal nonzero prime reflexive difference ideals. Let L be a primary, finitely-generated extension of K, $L = K\langle a_1, \ldots, a_n\rangle$. Then there is a difference valuation ring R of L dominating R_0.*

The proof is analogous to that in the differential case (see Theorem 2 of [5]), following here from Theorems 3 and 4.

THEOREM 6. *Let R_0 be a local difference ring with quotient field K. Let b be transformally independent over K and let $L = K\langle b, a_1, \ldots, a_n\rangle$ be a primary extension of $K\langle b\rangle$. Then there is a difference valuation ring of L dominating R_0.*

PROOF. In this case, the ideals $P_k = \{b - \tau^k b\}$, $k = 1, 2, \ldots$, provide a descending chain of prime reflexive ideals in $R\{b\}$, so that if $u \in R\{b\}$, $u \neq 0$, then

$u \notin P_k$ for some k. Then Theorem 4 yields a prime reflexive difference ideal in $R\{b, a_1, \ldots, a_n\}$ and the proof proceeds as in [5].

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