EXTENSIONS OF DIFFERENCE SPECIALIZATIONS

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Abstract. Maximal difference specializations and difference places are defined. Let \( R \) be the domain of a difference specialization \( \phi \) of a difference field \( K \) and \( x \in K \). Then \( \phi \) can be extended to a specialization \( x \to 0 \) if and only if \( 1 \notin \{x\} \). This result applies to give a condition on a polynomial for the extension of a specialization to its generic zero. In a slightly different direction, a necessary and sufficient condition for the extension of a specialization to a larger difference field is given.

Introduction. Numerous examples (see [3] and below) have shown that an extension of a difference specialization may be impossible while the corresponding algebraic extension is easy to obtain. When difference extensions exist remains a question of continued interest.

This work provides conditions for the extension of a specialization to one sending a particular element to 0 (Theorems 1 and 2). It is then possible to provide, in difference algebra, conditions for extensions of specializations to extension fields (Theorems 5 and 6), which are analogous to conditions given by S. D. Morrison in differential algebra [5].

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The basic definitions of difference algebra are assumed [3]. In a difference ring \( R \), the difference ideal generated by a set \( N \) is denoted by \( [N] \), while the perfect difference ideal is denoted \( \{N\} \). Let \( N^0 = N \), let \( N' = \{a \in R \mid \text{some power product of transforms of } a \text{ is in } N\} \) and let \( N^{(k+1)} = [N^{(k)}]' \), \( k = 1, 2, \ldots \). Then \( \{N\} = \bigcup_{k=0}^\infty N^{(k)} \). If \( R \) is an integral domain, then any difference homomorphism of \( R \) into a difference field is called a difference specialization of \( R \).

1. Difference places. Let \( K \) be a difference field with transforming operator \( \tau \). A maximal difference specialization of \( K \) is a difference homomorphism \( \phi \) of a difference subring of \( K \) onto a difference domain \( \Lambda \), which cannot be extended to a difference homomorphism of a larger difference subring of \( K \) onto a difference domain extension of \( \Lambda \). It may be noted that \( \Lambda \) is, in fact, a field. If \( \phi(x) \neq 0 \), then \( \phi(x^{-1}) \) can be defined by \( \phi(x)^{-1} \) in the field of quotients of \( \Lambda \), but \( \phi \) is maximal, so \( \phi(x)^{-1} \in \Lambda \). The domain \( R \) of \( \phi \) is called a maximal difference ring of \( K \). If \( K \) is the quotient field of \( R \), then \( R \) is called a difference valuation ring of \( K \) and \( \phi \) is called a difference place of \( K \). If \( K \) is an inversive difference field, then any difference homomorphism of a subring of \( K \) can be extended to its inversive closure in \( K \), so any maximal difference ring of an inversive field is inversive.

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Let $\phi$ be a maximal difference specialization with domain $R$ and let $M(R)$ denote the kernel of $\phi$. $M(R)$ is a prime reflexive difference ideal, which consists of the nonunits of $R$. Thus $R$ is a local ring with maximal ideal $M(R)$. More generally, a local difference ring is defined to be a difference ring whose nonunits form a difference ideal. Given any difference ring $R$ with prime difference ideal $P$, the local ring $R_P = \{r/s | s \notin P\}$ is a local difference ring if and only if $P$ is reflexive. This follows from the fact that $P$ is reflexive if and only if $s \notin P$ implies $s_1 \notin P$. Consequently, the maximal ideal $M(R)$ of a local difference ring is reflexive, since in this case $R = R_{M(R)}$.

For any local difference subring of a difference field $K$, the following are equivalent: (i) $R$ is a maximal difference ring of $K$; (ii) $R$ is maximal among local difference subrings of $K$, ordered by domination; (iii) if $x \in K$ and $x \notin R$, then $1 \in \{R \{x\} M(R)\}$, the perfect difference ideal generated by $M(R)$ in $R \{x\}$. The equivalence of (iii) follows from the fact that every proper perfect difference ideal is contained in a prime reflexive difference ideal.

Let $R$ a difference valuation ring of $K$. Then the set $U$ of units of $R$ forms a subgroup of $K^* = K - \{0\}$ and the natural homomorphism $\nu : K^* \to K^*/U$ may be defined. Let $K^*/U$ be denoted by $\Gamma$, with the operation written as addition. Then $\nu$ will be called a difference valuation of $K$. Let $\Gamma^+ = \nu(M(R)^*)$; then for $a \in \Gamma^+$, $-a \notin \Gamma^+$. For $a$ and $b \in \Gamma$, define $a < b$ if $b - a \notin \Gamma^+$. Then $\Gamma$ is an ordered group, but is not necessarily totally ordered. $x \in R^*$ if and only if $\nu(x) > 0$ and $x \in M(R)^*$ if and only if $\nu(x) > 0$.

The following example, due to R. M. Cohn, shows that there are difference valuation rings which are not valuation rings (and, thus, difference valuations which are not valuations). Let $C$ be the set of complex numbers and let $a$ be transcendental over $C$. Consider $C(a) = C\langle a \rangle$ as a difference field by defining $a_1 = -a$. Let $C\langle a \rangle \langle y \rangle$ be a difference polynomial ring and denote the transform of $y$ by $y_1$. Let $P = y^2 - y_1^2 + ay_1^2$. Then $P + P_1 = (y + y_2)(y - y_2)(1 + ay_1^2)$ and the variety $\mathcal{V}(P)$ has 2 components, one satisfying $y + y_2$ and the other satisfying $y - y_2$. Let $x$ be a generic zero of an irreducible component, with $x_2 = -x$. For any difference polynomial $A$, $x$ satisfies $A$ if and only if $x$ satisfies a first-order difference polynomial $B$ (obtained by substituting $-y$ for $y_2$), which is a multiple of $P$. It follows that $x$ specializes to 0 over $C\langle a \rangle$. This specialization can be extended to a maximal one and, hence, there is a difference valuation ring $R$ of $C\langle a, x \rangle$, with $x \in M(R)$. However, $R$ is not a valuation ring: $z = x_1/x$ is integral over $R$ since $z$ satisfies $1 - Z^2 + ax_1^2 \in R[Z]$, but $z \notin R$. If $\phi$ is the maximal specialization and $\phi(z)$ is defined, then $\phi(z^2) = \phi(1 + ax_1^2) = 1$ and $\phi(z_1) = 1$. But $zz_1 = x_1/x \cdot x_2/x_1 = -1$. Thus, $z$ is not in the domain of $\phi$.

2. Extensions of a specialization to an element. The following criterion provides a tool in specialization problems, as well as in the development of the theory of difference places.

**Theorem 1.** Let $R$ be a local difference subring of a difference field $K$ and let $x \in K$. The homomorphism $\phi : R \to R/M(R)$ extends to one sending $x$ to 0 if and only if $1 \notin [x]$ in $R \{x\}$.
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Proof. If \( \phi \) extends to a homomorphism \( \phi' \) of \( R(x) \) with \( \phi'(x) = 0 \), then \( [x] \subset \text{Ker}\ \phi' \). Therefore \( 1 \notin [x] \).

For the converse, let \( N \) denote the difference ideal generated by \( M(R) \) and \( x \) in \( R(x) \). Then \( N = R(x)M(R) + [x] = M(R) + [x] \). If \( \phi \) cannot be extended to a homomorphism of \( R(x) \) sending \( x \) to 0, then \( 1 \notin \{N\} \). It will be shown by an inductive argument that if there is \( c \in \{N\} \) with \( c = u + z, u \in U = R - M(R), \) \( z \in [x] \), then \( 1 \notin [x] \). Since \( 1 \in \{N\} \) and 1 is of this form (with \( z = 0 \)), the proof will be complete.

\[ \{N\} = \cup_{k=0}^{\infty} N^{(k)} \], so \( c \in [N^{(k)}] \) for some \( k \). If \( k = 0 \), then \( c \in N = M(R) + [x] \). Thus \( c = u + z = m + w \), where \( m \in M(R) \) and \( w \in [x] \). Since \( u \notin M(R) \), \( u - m \notin M(R) \) and \( (u - m)^{-1} \in R \). Then\( 1 = (u - m)^{-1}(u - m) - (w - z) \in [x] \). If \( k = n + 1 \), then \( c = \sum_{i=1}^{n} f_i q_i \), where \( f_i \in R \{x\} \) and \( q_i \in N^{(n+1)} \). For each \( i, f_i = r_i + z_i \) and \( q_i = s_i + y_i \), where \( r_i \) and \( s_i \) are in \( R \) and \( z_i \) and \( y_i \) are in \( [x] \). So \( \sum_{i=1}^{n} f_i q_i = \sum_{i=1}^{n} (r_i + z_i) s_i + y_i \) = \( t + w \), where \( t = \sum_{i=1}^{n} r_i s_i \) and \( w \in [x] \). Thus, \( c = u + z = t + w \). If \( t \in M(R) \), then, as above, \( (u - t)^{-1} \in R \) and \( 1 \notin [x] \). Otherwise, \( s_i \notin M(R) \) for some \( i \). For this \( i \), denote \( q_i = s_i + y_i \), simply by \( q = s + y \). Since \( q \in N^{(n+1)} \), there is a product \( \pi(q) = (s + y)^{q}(\tau s + \tau y)^{q} \cdots (\tau^{q} s^{q} + \tau^{q} y^{q}) \in [N^{(n)}] \). But \( \pi(q) = \bar{u} + \bar{z} \), where \( \bar{u} = s^{q}(\tau s)^{q} \cdots (\tau^{q} s^{q})^{q} \) and \( \bar{z} \in [x] \). Since \( s \notin M(R) \) and \( M(R) \) is prime and reflexive, \( \bar{u} \in R - M(R) \). Therefore, by induction on \( n \), \( 1 \notin [x] \).

Corollary. Let \( R \) be a maximal difference ring of \( K \), and let \( x \in K \). Then \( x \in M(R) \) if and only if \( 1 \notin [x] \).

Let \( K \{y\} \) be a difference polynomial ring with transform \( \tau \) and let \( \phi \) be a difference specialization of \( K \) with domain \( R \). Let \( g(y) \in K \{y\} \) and let \( x \) be a generic zero of a component of \( \mathcal{Q}(g) \). The specialization of the coefficients of \( g \) to 0 does not guarantee that \( \phi \) can be extended to a specialization of \( x \) to 0. R. M. Cohn has noted that even if \( g(y) \) is of the form \( y \tau y + b \), this may not be possible:

let \( b \) be a nonzero solution of the polynomial \( Q \) on p. 332 of [3]. However, a sufficient condition for such an extension can be given.

Let \( R \) be a difference subring of a difference field \( K \). Let \( g(y) \) be a difference polynomial in \( R \{y\} \). Let \( \{g\}_R \) and \( \{g\}_K \) be the perfect difference ideals generated by \( g \) in \( R \{y\} \) and \( K \{y\} \), respectively. The term of the polynomial which has no power of \( y \) or its transforms is called the constant term.

Theorem 2. Let \( g(y) \in R \{y\} \) have constant term \( b \in R \). Let \( \{g\}_K \) be prime and let \( x \) be a generic zero of \( \{g\}_K \). Let \( \phi: R \to \Lambda \) be a difference specialization of \( K \) with \( \phi(b) = 0 \). \( \phi \) can be extended to \( R \{x\} \) with \( \phi(x) = 0 \) if \( \{g\}_K \cap R \{y\} = \{g\}_R \).

Proof. It can be shown by induction that if \( f(y) \in \{g\}_R \) has constant term \( c \), then \( \phi(c) = 0 \). \( \{g\}_R = \cup_{k=0}^{\infty} \{g\}^{(k)} \), so \( f \in \{g\}^{(k)} \) for some \( k \). If \( k = 0 \), \( f(y) = \sum_{i} f_i(y) \tau^i g(y) \), \( f_i \in R \{y\} \). Comparison of constant terms yields \( c = \sum_{i} \tau^i b \), \( r_i \in R \), and hence \( \phi(c) = 0 \). If \( k = n + 1 \), then \( f(y) = \sum_{i} f_i(y) q_i(y) \), where \( f_i \in R \{y\} \) and a power product \( \pi(q_i) \in \{g\}^{(n)} \). If \( q_i \) has constant term \( a_i \), then \( \pi(q_i) \) has constant term \( \tau_i(a_i) \) and by induction, \( \phi(\pi(a_i)) = 0 \). Hence \( \phi(a_i) = 0 \) and \( \phi(c) = \phi(\sum_{i} a_i q_i) = 0 \). Consequently, if \( f(y) \in \{g\}_R \), its constant term \( \neq 1 \).
By extending \( \phi \) in \( K \), one may assume that \( R \) is a local difference subring of \( K \) and hence of \( K \{ x \} \), with \( \{ g \}_K \cap R \{ y \} = \{ g \}_R \). (For, if \( P = \text{Ker } \phi \) and \( R \) is replaced by \( S = R_P \), it follows that \( \{ g \}_K \cap S \{ y \} = \{ g \}_S \).) By Theorem 1, if \( \phi \) does not extend to \( x \to 0 \), then \( 1 \in \{ x \} \) in \( R \{ x \} \), i.e. \( 1 = \sum b_i(x)\tau'(x), \ b_i \in R \{ x \} \).

Then \( f(y) = 1 - \sum b_i(y)\tau'(y) \in \{ g \}_K \cap R \{ y \} \). Since \( f \) has constant term \( 1, f \notin \{ g \}_R \). Hence, \( \{ g \}_K \cap R \{ y \} \neq \{ g \}_R \).

The corollary to Theorem 1 applies to maximal difference rings, yielding the following development.

**Proposition 1.** Let \( R \) and \( S \) be maximal difference rings of \( K \) with \( R \subset S \). Then \( M(S) \subset M(R) \).

**Proof.** If \( x \in M(S) \), then \( 1 \notin [x] \cdot S \{ x \} \). But, since \( R \subset S \), \( 1 \notin [x] \cdot R \{ x \} \).

Hence \( x \in M(R) \).

It follows that if \( R \) and \( S \) are maximal difference rings of \( K \) with \( R \subset S \), then every ideal of \( S \) is an ideal of \( R \).

**Proposition 2.** Let \( R \) be a maximal difference ring of \( K \), and let \( P \) be a prime reflexive difference ideal of \( R \). Then there is a maximal difference ring \( S \) of \( K \) such that \( R \subset S \) and \( M(S) = P \).

**Proof.** Let \( S \) be a maximal local difference ring of \( K \), dominating \( R_P \). By Proposition 1, \( M(S) \subset M(R) \). Hence \( M(S) = M(S) \cap R = (M(S) \cap R_P) \cap R = P \cdot R_P \cap R = P \).

The next proposition follows from this with a proof similar to that given in the differential case in [5, Proposition 3].

**Proposition 3.** Let \( R \) be a maximal difference ring of \( K \). Then the prime reflexive difference ideals of \( R \) are linearly ordered by inclusion.

As a consequence, in a maximal difference ring, every perfect difference ideal is prime. The difference rank of a difference valuation ring is defined to be the number of prime reflexive ideals in the ring.

**Proposition 4.** Let \( S \) be a maximal difference ring of \( K \) with specialization \( \phi: S \to \Lambda \). If \( R \) is a maximal difference ring of \( K \) with \( R \subset S \), then \( \phi(R) \) is a maximal difference ring of \( \Lambda \).

**Proof.** Let \( \psi: R \to \Omega \) be a maximal specialization of \( K \) with domain \( R \). Since \( \text{Ker } \phi = M(S) \subset M(R) = \text{Ker } \psi, \ \theta: \phi(R) \to \Omega \) may be defined by \( \theta(\phi(r)) = \psi(r) \), with \( \text{Ker } \theta = \phi(M(R)) \). Let \( x \in \Lambda \); if \( x \notin \phi(R) \), there is \( s \in S, s \notin R \), such that \( \phi(s) = x \). Since \( \psi \) is maximal, \( 1 \in \{ R \{ s \} \cdot M(R) \} \) and thus, \( 1 \in \{ \phi(R) \{ x \} \cdot \text{Ker } \theta \} \). Therefore \( \theta \) cannot be extended to \( x \).

3. **Extensions of a place to an extension field.** If \( \phi \) is a difference place of \( K \) and \( L \) is a difference field extension of \( K \), there arises the question of whether \( \phi \) can be extended to a difference place of \( L \). A necessary and sufficient condition is provided in the corollary below. The development is analogous to that of the differential case [5].
Theorem 3. Let $R_0$ be a difference subring of $K$ with prime reflexive ideals $P$ and $Q$, $P \subseteq Q$ and let $S$ be a proper, maximal difference ring of $K$ with $R_0 \subseteq S$ and $M(S) \cap R_0 = P$. Then there is a proper maximal difference ring $R$ with $R_0 \subseteq R$ and $M(R) \cap R_0 = Q$. Furthermore, if $S$ is a difference valuation ring of $K$ then $R$ is also.

Proof. Similar to Theorem 1 [5].

Corollary 1. Let $R_0$ be a local difference subring of a difference field $K$. Let $L$ be a difference field extension of $K$ and let $S$ be proper maximal difference ring (valuation ring) of $L$ containing $R_0$. Then there is a proper maximal difference ring (valuation ring) $R$ of $L$ dominating $R_0$.

The existence of $S$ in the corollary is equivalent to the condition that $L$ have a subring $S_0$, $R_0 \subseteq S_0$, which contains a proper nonzero prime reflexive difference ideal. That the condition does not always hold is easily seen. For example, if $Q\{b\}$ is a difference ring with $b$ transcendental over the rationals $Q$ and $\sigma b = b$, the difference specialization $b \rightarrow 1$ does not extend to a difference place of $Q(\langle a \rangle)$, where $a^2 = b$ and $\sigma a = -a$. (See [3, Chapter 7, Example 3].) However, the condition does hold in the situations of Theorems 5 and 6. Theorem 4 is needed for these results.

Theorem 4. Let $R$ be a difference integral domain with quotient field $K$. If $K\langle a_1, \ldots, a_n \rangle$ is a difference field extension of $K$ which is a primary extension, then there is $u \in R$, $u \neq 0$, such that any specialization $\phi$ of $R$ with $\phi u \neq 0$ can be extended to $R\{a_1, \ldots, a_n\}$.

This theorem is a slight modification of part of Theorem IV of [3, Chapter 7] (where $R = F\{b_1, \ldots, b_m\}$, $F$ is a difference field and the specializations are over $F$). The generalization to an arbitrary domain $R$ can be obtained by using Proposition 9, Chapter 0 of [4] in the lemmas preceding the theorem.

Theorem 5. Let $R_0$ be a local difference ring with quotient field $K$ and no minimal nonzero prime reflexive difference ideals. Let $L$ be a primary, finitely-generated extension of $K$, $L = K\langle a_1, \ldots, a_n \rangle$. Then there is a difference valuation ring $R$ of $L$ dominating $R_0$.

The proof is analogous to that in the differential case (see Theorem 2 of [5]), following here from Theorems 3 and 4.

Theorem 6. Let $R_0$ be a local difference ring with quotient field $K$. Let $b$ be transformally independent over $K$ and let $L = K\langle b, a_1, \ldots, a_n \rangle$ be a primary extension of $K\langle b \rangle$. Then there is a difference valuation ring of $L$ dominating $R_0$.

Proof. In this case, the ideals $P_k = \langle b - \tau^k b \rangle$, $k = 1, 2, \ldots$, provide a descending chain of prime reflexive ideals in $R\{b\}$, so that if $u \in R\{b\}$, $u \neq 0$, then
Then Theorem 4 yields a prime reflexive difference ideal in $R\{b, a_1, \ldots, a_n\}$ and the proof proceeds as in [5].

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