LAGRANGE’S THEOREM WITH $N^{1/3}$ SQUARES

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Abstract. For every $N > 1$ we construct a set $A$ of squares such that $|A| < (4/\log 2)N^{1/3} \log N$ and every nonnegative integer $n < N$ is a sum of four squares belonging to $A$.

Let $A$ be an increasing sequence of nonnegative integers and let $A(x)$ denote the number of elements of $A$ not exceeding $x$. If every nonnegative integer up to $x$ is a sum of four elements of $A$, then $A(x)^4 > x$ and so $A(x) > x^{1/4}$. In 1770, Lagrange proved that every integer is a sum of four squares. If $A$ is a subsequence of the squares such that every nonnegative integer is a sum of four squares belonging to $A$, then we say that Lagrange’s theorem holds for $A$. Since there are $1 + [x^{1/2}]$ nonnegative squares not exceeding $x$, it is natural to look for subsequences $A$ of the squares such that Lagrange’s theorem holds for $A$ and $A$ is “thin” in the sense that $A(x) < cx^a$ for some $a < 1/2$.

Härtter and Zöllner [2] proved that there exist infinite sets $S$ of density zero such that Lagrange’s theorem holds for $A = \{n^2 \mid n \in S\}$. It is still true in this case that $A(x) \sim x^{1/2}$. Using probabilistic methods, Erdös and Nathanson [1] proved that, for every $\epsilon > 0$, Lagrange’s theorem holds for a sequence $A$ of squares satisfying $A(x) < cx^{(3/8)+\epsilon}$.

In this paper we study a finite version of Lagrange’s theorem. For every $N > 1$, we construct a set $A$ of squares such that $|A| < (4/\log 2)N^{1/3} \log N$ and every $n < N$ is the sum of four squares belonging to $A$. This improves the result of Erdös and Nathanson in the case of finite intervals of integers. We conjecture that for every $\epsilon > 0$ and $N > N(\epsilon)$ there exists a set $A$ of squares such that $|A| < N^{(1/4)+\epsilon}$ and every $n < N$ is the sum of four squares in $A$.

Let $|A|$ denote the cardinality of the finite set $A$ and let $[x]$ denote the greatest integer not exceeding $x$.

LemmA. Let $a > 1$. Let $n \equiv a^2$ and $n \equiv 0 \pmod{4}$. Then either $n - a^2$ or $n - (a - 1)^2$ is a sum of three squares.

Proof. If the positive integer $m$ is not a sum of three squares, then $m$ is of the form $m = 4^s(8t + 7)$. If $s = 0$, then $m \equiv 3 \pmod{4}$. If $s > 1$, then $m \equiv 0 \pmod{4}$.
Since $a - 1$, $a$ are two consecutive numbers, there exist $i, j \in \{0, 1\}$ such that $a - i$ is even and $a - j$ is odd, hence $(a - i)^2 \equiv 0 \pmod{4}$ and $(a - j)^2 \equiv 1 \pmod{4}$. If $n \equiv 1$ or $2 \pmod{4}$, then

$$n - (a - i)^2 \equiv n \equiv 1 \quad \text{or} \quad 2 \quad \pmod{4},$$

and so $n - (a - i)^2$ is a sum of three squares. If $n \equiv 3 \pmod{4}$, then

$$n - (a - j)^2 \equiv n - 1 \equiv 2 \quad \pmod{4},$$

and so $n - (a - j)^2$ is a sum of three squares. This proves the lemma.

**Theorem.** For every $N > 2$ there is a set $A$ of squares such that

$$|A| < \left(\frac{4}{\log 2}\right)^{N^{1/3}} \log N$$

and every nonnegative integer $n < N$ is a sum of four squares belonging to $A$.

**Proof.** Let $N > 6$. Let $A_1 = \{a^2|0 < a < 2N^{1/3} \text{ and } a^2 < N\}$ and let $A_2$ consist of the squares of all numbers of the form $\lfloor k^{1/2}N^{1/3} \rfloor - i$, where $4 < k < N^{1/3}$ and $i \in \{0, 1\}$. Then $|A_1| < 2N^{1/3} + 1$ and $|A_2| < 2N^{1/3} - 6$. Let $A_3 = A_1 \cup A_2$. Then $|A_3| < 4N^{1/3}$.

The set $A_1$ contains all squares not exceeding $\min(N, 4N^{2/3})$. Thus, if $0 < n < \min(N, 4N^{2/3})$, then $n$ is a sum of squares in $A_1 \subset A_3$. We shall show that if $4N^{2/3} < n < N$ and $n \equiv 0 \pmod{4}$, then there is an integer $b^2 \in A_2$ such that $0 < n - b^2 < 4N^{2/3}$ and $n - b^2$ is a sum of three squares. Since each of these squares does not exceed $4N^{2/3}$, it follows that $n - b^2$ is a sum of three squares in $A_3$, hence $n$ is a sum of four squares in $A_1 \cup A_2 = A_3$.

Suppose $4N^{2/3} < n < N$ and $n \equiv 0 \pmod{4}$. Let $k = \lfloor n/N^{2/3} \rfloor$. Then $4 < k < N^{1/3}$. Let $a = \lfloor k^{1/2}N^{1/3} \rfloor$. Then $a^2 < n$. Moreover, $a^2 \in A_2$ and $(a - 1)^2 \in A_2$. By the lemma, $n - b^2$ is a sum of three squares for either $b = a$ or $b = a - 1$. We must now show that $0 < n - b^2 < 4N^{2/3}$. Since $kN^{2/3} < n < (k + 1)N^{2/3}$ and $a < k^{1/2}N^{1/3} < a + 1$, it follows that

$$n - b^2 > n - a^2 > kN^{2/3} - (k^{1/2}N^{1/3})^2 = 0.$$  

Since $k < N^{1/3}$ and $4 < 3N^{1/6}$ for $N > 6$, it follows that

$$n - b^2 < (k + 1)N^{2/3} - (a - 1)^2$$

$$< (k + 1)N^{2/3} - (k^{1/2}N^{1/3} - 2)^2$$

$$< (k + 1)N^{2/3} - (kN^{2/3} - 4k^{1/2}N^{1/3})$$

$$= N^{2/3} + 4k^{1/2}N^{1/3}$$

$$< N^{2/3} + 4N^{1/2}$$

$$< 4N^{2/3}.$$  

Therefore, if $0 < n < N$ and $n \equiv 0 \pmod{4}$, then $n$ is a sum of four squares belonging to $A_3$.

Construct the finite set $A$ of squares as follows:

$$A = \{4^ra^2|a^2 \in A_3, r > 0, 4^ra^2 < N\}.$$
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Then $A_3 \subseteq A$ and

$$|A| < \left( \frac{\log N}{\log 4} + 1 \right)|A_3| < \left( \frac{2 \log N}{\log 4} \right)4N^{1/3}$$

$$= \left( \frac{4}{\log 2} \right)N^{1/3} \log N.$$

Let $0 < n < N$. Then $n = 4'm$, where $r > 0$ and $m \equiv 0 \pmod{4}$. Consequently, $m = a_1^2 + a_2^2 + a_3^2 + a_4^2$, where $a_i^2 \in A_3$. Then

$$n = 4'm = 4'a_1^2 + 4'a_2^2 + 4'a_3^2 + 4'a_4^2$$

$$= (2'a_1)^2 + (2'a_2)^2 + (2'a_3)^2 + (2'a_4)^2$$

is a sum of four squares in $A$. This proves the theorem for $N > 6$.

For $N < 6$, it suffices to consider the set $A = \{0, 1\}$ for $N = 2, 3$ and the set $A = \{0, 1, 4\}$ for $N = 4, 5$. This completes the proof.

REFERENCES


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