LOCAL ERGODIC THEOREMS FOR NONCOMMUTING SEMIGROUPS

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Abstract. Let \((X, \mu)\) be a \(\sigma\)-finite measure space and \(L^p(\mu), 1 < p < \infty,\) the usual Banach spaces of complex-valued functions. For \(k = 1, 2, \ldots, n,\) let \((T_k(t): t > 0)\) be a strongly continuous semigroup of Dunford-Schwartz operators. If

\[
f \in L_{n-1} = \left\{ g: \int_{|x| > t} \left| g/t \right| (\log |g/t|)^{n-1} \, d\mu < \infty \text{ for all } t > 0 \right\},
\]

then

\[
\frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n} \int_0^{\alpha_1} \cdots \int_0^{\alpha_n} T_n(t_n) \cdots T_1(t_1)f(x) \, dt_1 \cdots dt_n \rightarrow f(x)
\]

\(\mu\)-a.e. as \(\alpha_1 \nearrow 0, \ldots, \alpha_n \nearrow 0\) independently. If \(f \in L_p(\mu), 1 < p < \infty,\) then the limit exists in norm as well as pointwise.

Introduction. Let \((X, \mu)\) be a complete, \(\sigma\)-finite measure space and let \(L^p(\mu) = L^p(X, \mu), 1 < p < \infty,\) be the usual Banach spaces of complex-valued functions. Let \(\{T(t): t > 0\}\) be a strongly continuous semigroup of \(L^1(\mu)\) contractions. This means that (i) \(T(t + s) = T(t)T(s), s, t > 0;\) (ii) \(\|T(t)\|_1 < 1, t > 0;\) (iii) \(f \in L^1(\mu)\) implies \(\|T(t)f - T(s)f\|_1 \rightarrow 0\) as \(s \rightarrow t.\) We assume for simplicity that \(T(0) = I.\) A semigroup \(\{T(t)\}\) of \(L^1(\mu)\) contractions is a Dunford-Schwartz semigroup if \(\|T(t)\|_\infty < 1\) for all \(t > 0.\) It is a submarkovian semigroup if each \(T(t)\) is a positive operator, i.e. \(f \in L^+_1(\mu)\) implies \(T(t)f \in L^+_1(\mu)\) for all \(t > 0.\) A positive \(L^1(\mu)\) semigroup \(\{P(t)\}\) is said to dominate \(\{T(t)\}\) if \(P(t)|f| > |T(t)f|\) \(\mu\)-a.e. for \(f \in L^1(\mu)\) and \(t > 0.\)

The strong continuity of \(\{T(t)\}\) permits us to define, for \(\alpha > 0\) and \(f \in L^1(\mu),\) the integral \(\int_0^\infty T(t)f \, dt\) as the \(L^1\)-limit of Riemann sums. A more precise definition of \(\int_0^\infty T(t)f \, dt\) is required to investigate the pointwise convergence of \((1/\alpha)\int_0^\infty T(t)f \, dt\). It is well known ([2], [8]) that given \(f \in L^1(\mu)\) the vector \(T(t)f\) has a scalar representation \(T(t)f(x),\) defined on \(R^+ \times X\) and measurable with respect to the product measure on \(R^+ \times X,\) such that \(T(t)f(x)\) is in the equivalence class of \(T(t)f\) for all \(t > 0.\) This representation is unique modulo sets of product measure zero. The scalar function \(T(t)f(x)\) is integrable with respect to the product measure on \(R^+ \times X.\) Additionally, there is a \(\mu\)-null set \(E(f),\) independent of \(\alpha > 0,\) outside of which \(\int_0^\infty T(t)f(x) \, dt\) exists and, as a function of \(x,\) is in the equivalence class of \(\int_0^\infty T(t)f \, dt\) for every \(\alpha > 0.\) We define
\[ A(T, \alpha)f(x) = \frac{1}{\alpha} \int_0^\alpha T(t)f(x) \, dt \]

for all \(\alpha > 0\) and \(f \in L_1(\mu)\).

In [3] N. Fava showed that if \(f \in R_{n-1}\) and \(\{T_k(t): t > 0\}, k = 1, 2, \ldots, n\), are strongly continuous semigroups of positive Dunford-Schwartz operators, then

\[ \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n} \int_0^{\alpha_1} \cdots \int_0^{\alpha_n} T_n(t_n) \cdots T_1(t_1)f(x) \, dt_1 \cdots dt_n \]

converges \(\mu\)-a.e. to a finite limit as \(\alpha_1 \to \infty, \ldots, \alpha_n \to \infty\) independently. The class

\[ R_n = \left\{ f: \int_{|f|>1} |f|/t (|\log|f|/t|)^n \, d\mu < \infty \text{ for all } t > 0 \right\} \]

is a subspace of \(L_1(\mu) + L_\infty(\mu)\) and satisfies \(L_1(\mu) + L_\infty(\mu) \supseteq R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots\). Also, for any \(1 < p < \infty\) and \(n > 0\), \(L_p(\mu) \subseteq R_n\). Finally, \(R_n = L(\log^+ L)^n\), for all \(n > 0\), when \(\mu(X) < \infty\). These facts are established in [3].

In this note a local ergodic theorem is established: if \(\{T_k(t)\}, k = 1, 2, \ldots, n\), are strongly continuous semigroups of Dunford-Schwartz operators and \(f \in R_{n-1}\), then

\[ \frac{1}{\alpha_1 \cdots \alpha_n} \int_0^{\alpha_n} \cdots \int_0^{\alpha_1} T_n(t_n) \cdots T_1(t_1)f(x) \, dt_1 \cdots dt_n \to f(x) \quad \mu\text{-a.e.} \quad (\ast) \]

as \(\alpha_1 \to 0, \ldots, \alpha_n \to 0\) independently. For notational convenience we denote the integral in \((\ast)\) by

\[ A(T_1, \alpha_1) \cdots A(T_n, \alpha_n)f(x) \]

Local ergodic theorems for single \(L_1\) contraction semigroups have been established in ([4], [5], [6], [8], [9]). T. Terrell [10] extended the local ergodic theorem for one-parameter submarkovian semigroups to the \(n\)-parameter case. He showed that if \(f \in L_1(\mu)\) then

\[ \lim_{\alpha \to 0} \left(1/\alpha\right)^n \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_n)f(x) \, dt_1 \cdots dt_n = f(x) \quad \mu\text{-a.e.} \]

He pointed out that if it is assumed only that \(f \in L_1(\mu)\) then \((\ast)\) may fail (even if the semigroups commute).

**Main results.** If \(\{T(t): t > 0\}\) is a strongly continuous \(L_1(\mu)\) semigroup of Dunford-Schwartz operators then a scalar representation \(T(t)f(x)\) exists for any \(f \in L_p(\mu), 1 < p < \infty\) ([2, pp. 196–198]). However \(R_n\) is not contained in the linear span of \(\bigcup_{1 < p < \infty} L_p(\mu)\) [3]. Before proving our ergodic theorem we must show that a scalar representation exists for functions in \(R_0\).

**1. Lemma.** Let \((X, \mu)\) be a complete, \(\sigma\)-finite measure space and let \(\{T(t)\}\) be a strongly continuous \(L_1(\mu)\) contraction semigroup such that for all \(t > 0\), \(\|T(t)f\|_{\infty} < \|f\|_{\infty}\), \(f \in L_1(\mu) \cap L_\infty(\mu)\). Then \(\{T(t)\}\) may be extended to a Dunford-Schwartz semigroup and the domain of definition of the scalar representation of \(\{T(t)\}\) may be extended from \(L_1(\mu)\) to \(L_1(\mu) + L_\infty(\mu)\).

**Proof.** Let \(\{P(t): t > 0\}\) be a strongly continuous submarkovian semigroup which dominates \(\{T(t): t > 0\}\) ([4], [6]). If \(f \in L_1^+(\mu)\) then \(P: f \to e^{-t}P(t)f(x)\) defines a linear contraction mapping from \(L_1(\mu)\) to \(L_1(R^+ \times X, \nu)\), where \(\nu = dt \times d\mu\). This mapping is a positive contraction since if \(f > 0\) \(\mu\)-a.e. then \(P(t)f > 0\).
μ-a.e. for all \( t > 0 \) implies \( Pf > 0 \) p-a.e. by Fubini’s theorem. Consequently \( P \) may be extended to the cone of positive measurable functions on \( X \) [7]. We denote the extension of \( P \) by \( \hat{P} \).

Choose \( f \in L^+_\infty(\mu) \) and let \( f_k (k = 1, 2, \ldots) \) be a sequence of functions in \( L^1_\infty(\mu) \cap L^+_\infty(\mu) \) with \( f_k(x) \nearrow f(x) \) μ-a.e. as \( k \to \infty \). Then \( \hat{P}f(t, x) = \lim_{k \to \infty} e^{-tp}f_k(x), \) since \( \hat{P} \) has the monotone continuity property. We have \( |\hat{P}f(t, x)| < \infty \) μ-a.e. since \( \|\hat{P}g\|_\infty < \|g\|_\infty \) for all \( g \in L^+_\infty(\mu) \). Consequently the sequence \( \{T(t)f_k(x)\} \) is Cauchy μ-a.e. since

\[
|T(t)f_{k+j}(x) - T(t)f_k(x)| < |P(t)f_{k+j}(x) - P(t)f_k(x)|.
\]

Set \( \hat{T}(t)f(x) = \lim_{k \to \infty} T(t)f_k(x), f \in L^+_\infty(\mu) \). It is not difficult to show that our definition is independent, modulo \( \mu \)-null sets, of the particular sequence \( \{f_k\} \) converging to \( f \). Extend now \( \hat{T}(t)f(x) \) to \( L^\infty(\mu) \) by linearity. If we set \( T(t)f = \hat{T}(t)f(\cdot), f \in L^\infty(\mu) \), then we have extended \( \{T(t)\} \) to a Dunford-Schwartz semigroup and \( \hat{T}(t)f(x) \) is in the equivalence class of \( T(t)f \) for all \( t > 0 \) and \( f \in L^\infty(\mu) \).

One can see that given \( f \in L^\infty(\mu) \), there exists a \( \mu \)-null set \( E(f) \), independent of \( \alpha > 0 \), outside of which \( \int_0^t \hat{T}(t)f(x) \, dt \) exists and is finite. If \( f \in L^1(\mu) \cap L^\infty(\mu) \) then \( \hat{T}(t)f(x) \) and \( T(t)f(x) \) are equivalent scalar representations of \( T(t)f \). Finally, if \( f = g + h \), where \( g \in L^1(\mu) \) and \( h \in L^\infty(\mu) \), we define

\[
T(t)f(x) = \hat{T}(t)g(x) + \hat{T}(t)h(x).
\]

We note that this definition of \( T(t)f(x) \) is independent, modulo a \( \mu \)-null set, of the particular \( g \) and \( h \) chosen for the representation of \( f \). □

2. Theorem. Let \((X, \mu)\) be a complete, σ-finite measure space and let \( \{T_k(t): t > 0\} \), \( k = 1, 2, \ldots, n \), be strongly continuous \( L^1(\mu) \) contraction semigroups such that, for all \( t > 0 \) and \( f \in L^1(\mu) \cap L^\infty(\mu) \), \( \|T_k(t)f\|_\infty < \|f\|_\infty \). If \( f \in L^p(\mu) \), \( 1 < p < \infty \), then \( A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)f(x) \to f(x) \) both in norm and pointwise as \( \alpha_1 \searrow 0, \ldots, \alpha_n \searrow 0 \) independently.

Proof. By Lemma 1 each \( \{T_k(t)\} \) may be regarded as a Dunford-Schwartz semigroup. As pointed out in [2], \( \{T_k(t)\} \) is a strongly continuous subgroup of \( L^p(\mu) \) contractions for \( 1 < p < \infty \). If \( f \in L^p(\mu) \) and if \( f^p_<(x) \) denotes \( \sup_{\alpha > 0}|(1/\alpha)|T_k(t)f(x) \, dt| \), then \( \|f^p_<(\cdot)\|_p < (p/(p - 1))\|f\|_p \) by Theorem VIII.7.7 in [2]. Since \( L^1(\mu) \cap L^p(\mu) \) is dense in \( L^p(\mu) \) and \( \lim_{\alpha \searrow 0} A(T_k, \alpha)f(x) = f(x) \) μ-a.e. for all \( f \in L^1(\mu) \cap L^p(\mu) \) by Ornstein’s theorem [8, p. 108], it follows from Banach’s convergence principle [2, Theorem IV.11.3] that, for each \( k \), \( \lim_{\alpha \searrow 0} A(T_k, \alpha)f(x) \) exists and is finite μ-a.e. as \( \alpha \searrow 0 \) through some countable set, say \( Q^+ \), the set of positive rationals. Since \( A(T_k, \alpha_k)f(x) \) depends continuously on \( \alpha_k \) for a.e. \( x \), we have \( \lim_{\alpha \searrow 0} A(T_k, \alpha_k)f(x) \) exists and is finite μ-a.e. for every \( f \in L^p(\mu) \). The fact that \( \lim_{\alpha \searrow 0} A(T_k, \alpha_k)f(x) = f(x) \) μ-a.e. follows from the strong continuity of \( \{T(t)\} \) at \( t = 0 \).
Now consider the convergence of \( A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x) \). Since
\[
|A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x) - f(x)|
\leq \sum_{k=1}^{n-1} |A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)]| + |A(T_n, \alpha_n) f(x) - f(x)|,
\]
our result is established if we can show
\[
A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)] \to 0 \quad \text{\( \mu \)-a.e.}
\]
for \( k = 1, 2, \ldots, n - 1 \). If \( \{P_1(t)\}, \ldots, \{P_k(t)\} \) are strongly continuous semigroups of positive Dunford-Schwartz operators which dominate, respectively, \( \{T_1(t)\}, \ldots, \{T_k(t)\} \), then
\[
|A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)]| < A(P_n, \alpha_n) \cdots A(P_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)]\] for any \( \alpha_k > 0 \).

Consequently, given \( \varepsilon > 0 \),
\[
\mu \left\{ \limsup_{\alpha_n \to 0, \ldots, \alpha_1 \to 0} |A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)]| > \varepsilon \right\} \leq \left( \frac{1}{\varepsilon} \right)^p \left( \frac{p}{p-1} \right)^p \|A(T_k, \alpha_k) f(\cdot) - f\|_p^p
\]
for any \( \alpha_k > 0 \).

Since \( \|A(T_k, \alpha_k) f - f\|_p \to 0 \) as \( \alpha_k \to 0 \) by dominated convergence, we have
\[
A(T_n, \alpha_n) \cdots A(T_{k+1}, \alpha_{k+1}) [A(T_k, \alpha_k) f(x) - f(x)] \to 0
\]
\( \mu \)-a.e. as \( \alpha_1 \to 0, \ldots, \alpha_n \to 0 \) independently. The norm convergence of \( A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f \) follows from dominated convergence. \( \square \)

3. Theorem. Let \( \{T_k(t)\}, k = 1, 2, \ldots, n \), be strongly continuous \( L_1(\mu) \) contraction semigroups such that \( \|T_k(t)f\|_\infty < \|f\|_\infty, f \in L_1(\mu) \cap L_\infty(\mu) \). Then
\[
\lim_{\alpha_n \to 0, \ldots, \alpha_1 \to 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x) = f(x) \quad \text{\( \mu \)-a.e.}
\]
for \( f \in \mathcal{R}_{n-1} \).

Proof. For \( f \in \mathcal{R}_{n-1} \), set
\[
\omega(f) = \limsup_{\alpha_n \to 0, \ldots, \alpha_1 \to 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x) = - \liminf_{\alpha_n \to 0, \ldots, \alpha_1 \to 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x).
\]
Choose a sequence \( f_k (k = 1, 2, \ldots) \) of functions in \( L_p(\mu) \), \( 1 < p < \infty \), with \( f_k(x) \to f(x) \) \( \mu \)-a.e. and \( |f - f_k| < |f| \) for all \( k \). Then
\[
\omega(f) < \omega(f - f_k) + \omega(f_k) < \omega(f - f_k)
\]
by the preceding theorem. So \( \omega(f) < \omega(f - f_k) < 2(f - f_k)^* \), where

\[
(f - f_k)^*(x) = \sup_{\alpha_1 > 0, \ldots, \alpha_n > 0} |A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)(f - f_k)(x)|.
\]

Since the semigroups \( \{ T_i(t) \} \) are dominated by positive Dunford-Schwartz semigroups, it follows from Fava’s dominated estimate [3] that

\[
\mu\{ \omega(f) > 8t \} < \mu\{ (f - f_k)^* > 4t \} < C_n^{-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left( \log \frac{|f - f_k|}{t} \right)^{n-1} d\mu \text{ for any } t > 0.
\]

The integral approaches zero as \( k \to \infty \) by dominated convergence. Thus \( \mu\{ \omega(f) > 8t \} = 0 \) for all \( t > 0 \), and so \( \lim_{n \to 0} A(T_n, \alpha_n) A(T_{n-1}, \alpha_{n-1}) \cdots A(T_1, \alpha_1) f(x) \) exists and is finite \( \mu \)-a.e. Setting

\[
\tilde{f}(x) = \lim_{\alpha_1 > 0, \ldots, \alpha_n > 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1) f(x),
\]

we have

\[
\mu\{ (\tilde{f} - f_k) > 4t \} = \mu\{ \lim_{\alpha_1 > 0, \ldots, \alpha_n > 0} A(T_n, \alpha_n) \cdots A(T_1, \alpha_1)(f - f_k) > 4t \}
\]

\[
< \mu\{ (f - f_k)^* > 4t \}
\]

\[
< C_n^{-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left( \log \frac{|f - f_k|}{t} \right)^{n-1} d\mu.
\]

Likewise

\[
\mu\{ (f_k - \tilde{f}) > 4t \} < C_n^{-1} \int_{|f - f_k| > t} \left| \frac{f - f_k}{t} \right| \left( \log \frac{|f - f_k|}{t} \right)^{n-1} d\mu.
\]

These two inequalities imply \( f_k \to \tilde{f} \) in \( \mu \)-measure. By Corollary III.6.13 in [2], there exists a subsequence \( \{ f_{k_n} \} \) which converges to \( \tilde{f} \) \( \mu \)-a.e. Since \( f_k \to f \) pointwise, we must have \( \tilde{f}(x) = f(x) \) \( \mu \)-a.e. □

Since \( L_1(\mu) \) is strictly contained in \( R_0 \), the preceding theorem for the case \( n = 1 \) yields a slight generalization of Ornstein’s local ergodic theorem [8].

REFERENCES