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BRANCHING AND GENERALIZED-RECURSIVE INSET ENTROPIES

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ABSTRACT. In the framework of the mixed theory of information, the general form of branching entropies is determined. This result then leads to a characterization of regular, generalized-recursive entropies of randomized systems of events.

1. Introduction. A series of papers ([3], [1], [4], [8], [9]) by Aczél, Daróczy, and Kannappan has initiated the study of a new, mixed (as distinguished from probabilistic and nonprobabilistic) theory of information. Therein, entropies (and other measures of information) depend not only upon the probabilities of the events, but also directly upon the events themselves.

Let X be a ring of sets (containing, with any two sets, their union and difference, hence also their intersection and the empty set 0). A sequence of maps I_n: $X^n \times p(X)^n \to R$ ($n = 2, 3, \dots$; R the set of reals; p: $X \to [0, 1]$ a probability measure) will be called an *entropy of randomized systems of events*, or simply *inset entropy*. We seek inset entropies which satisfy certain desirable properties, analogous to those in the probabilistic theory of information (cf. Aczél-Daróczy [2]).

In the probabilistic theory, C. T. Ng [7] has characterized branching, symmetric entropies as exactly those which are representable as a sum. A similar result has been proven by the author [5] for entropies of sequences of elements of a semigroup. In the present setting of the mixed theory of information, an inset entropy I_n ($n = 2, 3, \dots$) is said to be *branching* if there exist maps $\Delta_{n,i}$: $X^3 \times p(X)^3 \to R$ ($i = 1, 2, \dots, n-1$) such that, for all $(x_1, x_2, \dots, x_n) \in X^n$,

$$I_n\begin{pmatrix} x_1, & x_2, \dots, & x_n \\ p(x_1), & p(x_2), \dots, & p(x_n) \end{pmatrix}$$
$$= I_n\begin{pmatrix} x_1, \dots, x_{i-1}, & x_i \cup x_{i+1}, & 0, & x_{i+2}, \dots, x_n \\ p(x_1), \dots, p(x_{i-1}), & p(x_i \cup x_{i+1}), & 0, & p(x_{i+2}), \dots, p(x_n) \end{pmatrix}$$
$$+ \Delta_{n,i}\begin{pmatrix} x_i, & x_{i+1}, & x_i \cup x_{i+1} \\ p(x_i), & p(x_{i+1}), & p(x_i \cup x_{i+1}) \end{pmatrix}. \tag{1.1}$$

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THEOREM 1.1. *I_n: $X^n \times p(X)^n \to R$ ($n = 2, 3, \ldots$) is (1.1) branching if, and only if, there exist maps $\phi_{n,i}$: $X \times p(X) \to R$ ($i = 0, 1, \ldots, n$), such that, for all $(x_1, x_2, \ldots, x_n) \in X^n$,*

$$I_n\begin{pmatrix} x_1, & x_2, \ldots, & x_n \\ p(x_1), & p(x_2), \ldots, & p(x_n) \end{pmatrix}$$

$$= \sum_{i=1}^{n} \phi_{n,i}(x_i, p(x_i)) + \phi_{n,0}\left(\bigcup_{i=1}^{n} x_i, p\left(\bigcup_{i=1}^{n} x_i\right)\right),$$

$$\phi_{n,i}(0, 0) = 0 \quad \textit{for } i = 1, 2, \ldots, n. \tag{1.2}$$

PROOF. Define a binary operation $*$: $[X \times p(X)]^2 \to R$ by

$$(x, p(x)) * (y, p(y)) := (x \cup y, p(x \cup y))$$

for all $x, y \in X$. Since $(X \times p(X), *)$ is an idempotent, abelian monoid (with unity $(0, 0)$), Theorem 3.2 of [5] gives the complete solution of (1.1) as (1.2).

We now examine a special form of (1.1) which I_n may satisfy if the x_i's are disjoint. Let

$$X_n := \{(x_1, x_2, \ldots, x_n) \in X^n | x_i \cap x_j = 0 \text{ for } i \neq j;\ i, j = 1, 2, \ldots, n\}$$

and

$$\Gamma_n := \left\{(p_1, p_2, \ldots, p_n) | \sum_{i=1}^{n} p_i = 1;\ p_i > 0 \text{ for } i = 1, 2, \ldots, n\right\}$$

for all $n = 2, 3, \ldots$. An inset entropy I_n ($n = 2, 3, \ldots$) is said to be *generalized-recursive* if

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = I_{n-1}\begin{pmatrix} x_1 \cup x_2, x_3, \ldots, x_n \\ p_1 + p_2, p_3, \ldots, p_n \end{pmatrix}$$

$$+ g(p_1 + p_2) I_2\begin{pmatrix} x_1, & x_2 \\ p_1/p_1 + p_2, & p_2/p_1 + p_2 \end{pmatrix} \tag{1.3}$$

whenever $p_1 + p_2 > 0$, for all

$$\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} \in X_n \times \Gamma_n \qquad (n = 2, 3, \ldots),$$

where $p_i = p(x_i)$ and g: $]0, 1] \to R$ satisfies

$$g(pq) = g(p)g(q), \qquad \forall p, q \in]0, 1]. \tag{1.4}$$

This property generalizes both the *recursivity* (in which $g(p) \equiv p$) introduced in [3] and the *α-recursivity* (in which $g(p) \equiv p^\alpha$) introduced in [4].

Generalizing a probabilistic result of the author [6], all regular, symmetric, expansible, branching entropies which are also generalized-recursive are determined. In the probabilistic setting, the entropies determined are essentially the Shannon entropy, defined on Γ_n by

$$H_n(p_1, p_2, \ldots, p_n) := -\sum_{i=1}^{n} p_i \log_2 p_i \qquad (0 \log_2 0 := 0), \tag{1.5}$$

and entropies of degree $\alpha \neq 1$,

$$H_n^\alpha(p_1, p_2, \ldots, p_n) := (2^{1-\alpha} - 1)^{-1}\left(\sum_{i=1}^n p_i^\alpha - 1\right) \qquad (0^\alpha := 0), \tag{1.6}$$

$(p_1, p_2, \ldots, p_n) \in \Gamma_n$; $n = 2, 3, \ldots$. We shall see these again as parts of generalized-recursive inset entropies.

REMARK 1.2. In [3], [1], [4], inset entropies are defined only on $X_n \times \Gamma_n$. If entropies are to be defined only on this restricted domain, then representation (1.2) can be assumed initially (on $X_n \times \Gamma_n$) instead of the (1.1) branching property. Of course it would be desirable to have a Theorem 1.1 on $X_n \times \Gamma_n$, but this seems to be very difficult. Note also that the p_i are independent of the x_i; the notations $p(x_i)$, $p(x)$ are used initially partly because $\sum_{i=1}^n p(x_i) \geqslant 1$ is possible.

2. Main result. First, we have an easy consequence of Theorem 1.1. An entropy I_n $(n = 2, 3, \ldots)$ is *symmetric* if, for all permutations π on $(1, 2, \ldots, n)$ and all $\binom{x_i}{p_i}_i \in X^n \times [0, 1]^n$,

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = I_n\begin{pmatrix} x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)} \\ p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)} \end{pmatrix}, \tag{2.1}$$

and *expansible* if, for all $\binom{x_i}{p_i}_i \in X^n \times [0, 1]^n$,

$$I_{n+1}\begin{pmatrix} x_1, x_2, \ldots, x_n, 0 \\ p_1, p_2, \ldots, p_n, 0 \end{pmatrix} = I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix}. \tag{2.2}$$

COROLLARY 2.1. *I_n: $X^n \times [0, 1]^n \to R$ $(n = 2, 3, \ldots)$ is* (1.1) *branching,* (2.1) *symmetric, and* (2.2) *expansible if, and only if, there exist maps ϕ, ϕ_0: $X \times [0, 1] \to R$ such that, for all $\binom{x_i}{p_i}_i \in X^n \times [0, 1]^n$,*

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \sum_{i=1}^n \phi(x_i, p_i) + \phi_0\left(\bigcup_{i=1}^n x_i, p\left(\bigcup_{i=1}^n x_i\right)\right), \tag{2.3}$$

$$\phi(0, 0) = 0. \tag{2.3A}$$

PROOF. Theorem 1.1 and [5, Corollary 7.5].

In view of Corollary 2.1, we state the principal result as follows.

THEOREM 2.2. *An inset entropy I_n $(n = 2, 3, \ldots)$ is* ((1.3), (1.4)) *generalized-recursive and has the* (2.3) *sum representation on $X_n \times \Gamma_n$ with* (2.3A) *and*

$$p \mapsto \phi(x, p) \in \mathfrak{L}^1]0, 1[\qquad (\forall x \in X)$$

if, and only if, it is representable, for all $\binom{x_i}{p_i}_i \in X_n \times \Gamma_n$, in one of the forms

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \gamma H_n(p_1, p_2, \ldots, p_n) + \sum_{i=1}^n p_i b(x_i) - b\left(\bigcup_{i=1}^n x_i\right), \tag{2.4}$$

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \sum_{i=1}^n p_i^\alpha a(x_i) - a\left(\bigcup_{i=1}^n x_i\right) \qquad (\alpha \neq 0, 1), \tag{2.5}$$

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \gamma + f\left(\bigcup_{i=1}^n x_i\right), \tag{2.6}$$

$$I_n\begin{pmatrix} x_1, x_2, \dots, x_n \\ p_1, p_2, \dots, p_n \end{pmatrix} = \sum_{p_i \neq 0} f(x_i) + \sum_{p_i = 0} h(x_i) - f\left(\bigcup_{i=1}^{n} x_i\right), \tag{2.7}$$

where γ is a constant, $a, b, f, h: X \to R$ with $h(0) = 0$ and H_n is defined by (1.5).

Note that (2.4), (2.5) are the same forms found in **[3]**, **[4]**, respectively.

The "if" part of Theorem 2.2 is a simple verification. For the "only if" part, we start with

LEMMA 2.3. *If I_n is generalized-recursive and has the* (2.3) *sum representation with $p \mapsto \phi(x, p) \in \mathfrak{L}^1]0, 1[$, then ϕ has one of the following four forms* (*with corresponding g of* (1.3)).

$$\phi(x,p) = a(x)p \log_2 p + b(x)p - c(x), \qquad g(p) = p, \tag{2.8}$$

$$\phi(x,p) = a(x)p^\alpha + b(x)p - c(x), \qquad g(p) = p^\alpha \qquad (\alpha \neq 1), \tag{2.9}$$

$$\phi(x,p) = a(x)p - c(x), \qquad g = \textit{arbitrary solution of } (1.4) \tag{2.10}$$

for all $p \in]0, 1]$ and all $x \in X$. Furthermore, provided $g \not\equiv 0$, the ϕ_0 of (2.3) *satisfies*

$$\phi_0(x, 1) = -\phi(x, 1), \qquad \forall x \in X. \tag{2.11}$$

PROOF. Substituting (2.3) for $\binom{x_i}{p_i}_i \in X_n \times \Gamma_n$ into (1.3), we have (with $p = p_1$, $q = p_2$),

$$\begin{aligned} &\phi(x_1, p) + \phi(x_2, q) - \phi(x_1 \cup x_2, p + q) \\ &\quad = g(p + q)\left[\phi(x_1, p/p + q) + \phi(x_2, q/p + q) + \phi_0(x_1 \cup x_2, 1)\right], \end{aligned} \tag{2.12}$$

for all $(p, q) \in D := \{(p, q) | p, q \geq 0, 0 < p + q \leq 1\}$. Fixing $(x_1, x_2) \in X_2$ temporarily and defining $h_1, h_2, h_3: X \to R$ and $\beta \in R$ by

$$h_i(p) := \phi(x_i, p), \qquad i = 1, 2, \qquad h_3(p) := \phi(x_1 \cup x_2, p), \tag{2.13}$$
$$\beta := \phi_0(x_1 \cup x_2, 1),$$

we obtain from (2.12), for all $(p, q) \in D$,

$$h_1(p) + h_2(q) - h_3(p + q) = g(p + q)\left[h_1(p/p + q) + h_2(q/p + q) + \beta\right]. \tag{2.14}$$

With $p = 0$, resp. $q = 0$, (2.14) yields

$$h_2(q) = h_3(q) + \delta_1 g(q) + \delta_2, \qquad h_1(p) = h_3(p) + \delta_3 g(p) + \delta_4, \tag{2.15}$$

for some constants δ_i ($i = 1, 2, 3, 4$). Now (2.14), (2.15), and the (1.4) multiplicativity of g give, for all $(p, q) \in D' := \{(p, q) | (p, q) \in D, pq \neq 0\}$,

$$\begin{aligned} &h_3(p) + h_3(q) - h_3(p + q) + \delta_2 + \delta_4 \\ &\quad = g(p + q)\left[h_3(p/p + q) + h_3(q/p + q) + \delta_2 + \delta_4 + \beta\right]. \end{aligned} \tag{2.16}$$

Defining F: $[0, 1] \to R$ by

$$F(p) := \begin{cases} 0, & \text{if } p = 0, \\ h_3(p) + (\beta - \delta_2 - \delta_4)p + \delta_2 + \delta_4, & \text{if } p \in]0, 1], \end{cases} \tag{2.17}$$

we can write (2.16) as

$$F(p) + F(q) - F(p + q) = g(p + q)\left[F(p/p + q) + F(q/p + q)\right]. \tag{2.18}$$

So far, (2.18) holds only on D', but we need it for all $(p, q) \in D$.

To this end, put $q = 1 - p$ in (2.12). If $g(1) = 1$, then (2.11) holds. On the other hand, if $g(1) \neq 1$, then $g \equiv 0$ by (1.4) with $q = 1$. Now let us check that (2.18) is true for, say, $p = 0$. It is true, by (2.17) and (2.13), if $0 = g(q)F(1) = g(q)[h_3(1) + \beta] = g(q)[\phi(x_1 \cup x_2, 1) + \phi_0(x_1 \cup x_2, 1)]$. But this is true since either $g \equiv 0$ or (2.11) holds. Thus we have (2.18) for all $(p, q) \in D$.

Furthermore, by the integrability assumption on ϕ, and by (2.13) and (2.17), $F \in \mathfrak{L}^1]0, 1[$. Under this condition, the only solutions of (2.18) (with (1.4)) are given by the author [6] as

$$F(p) = ap \log_2 p, \qquad g(p) = p \qquad (0 \log_2 0 := 0);$$

$$F(p) = a(p^\alpha - p), \qquad g(p) = p^\alpha \qquad (0^\alpha := 0, \alpha \neq 1);$$

$$F(p) = ap, \qquad g(p) = 0;$$

$$F(p) = \begin{cases} k, & p = 0, \\ a(1-p), & p > 0, \end{cases} \qquad g(p) = 1;$$

$$F(p) = 0, \qquad g = \text{arbitrary solution of (1.4)};$$

for some constants a, k, and α ($\neq 1$). Comparison of these results with (2.17) yields, for suitable constants a, b, c, and only for $p \in]0, 1]$,

$$h_3(p) = ap \log_2 p + bp - c, \qquad g(p) = p;$$

$$h_3(p) = ap^\alpha + bp - c, \qquad g(p) = p^\alpha \qquad (\alpha \neq 1);$$

$$h_3(p) = ap - c, \qquad g = \text{arbitrary solution of (1.4)}.$$

In view of (2.13), letting $(x_1, x_2) \in X_2$ vary, the formulas for h_3 and g give rise to solutions (2.8) through (2.10) for ϕ and g.

3. Proof of the main result. Theorem 2.2 is established by Lemma 2.3 in conjunction with the three lemmas of this section.

LEMMA 3.1. *If I_n ($n = 2, 3, \ldots$) is generalized-recursive and has the* (2.3) *sum form on $X_n \times \Gamma_n$, with ϕ, ϕ_0, g given by* (2.8), (2.11), *then I_n has the form* (2.4) *for some constant γ and map b: $X \to R$.*

PROOF. Substituting (2.8) into (2.12), and using (2.11), we get, after some rearrangement,

$$\left[c(x_1) + c(x_2) - c(x_1 \cup x_2)\right](p + q - 1)$$
$$= \left[a(x_1 \cup x_2)(p + q) - a(x_1)p - a(x_2)q\right]\log_2(p + q), \tag{3.1}$$

for all $(p, q) \in D'$ and all $(x_1, x_2) \in X_2$. Examining terms which are constant (with respect to p and q) in (3.1), we see that

$$c(x_1 \cup x_2) = c(x_1) + c(x_2), \qquad \forall (x_1, x_2) \in X_2. \tag{3.2}$$

Now (3.1) gives

$$a(x_1 \cup x_2)(p + q) = a(x_1)p + a(x_2)q,$$

which means that, for $\gamma := -a(0)$,

$$a(x) = -\gamma, \qquad \forall x \in X. \tag{3.3}$$

Now (2.12) with $p = 0 \neq q$ yields (by (2.8), (3.2), and (3.3)) $\phi(x_1, 0) + c(x_1) = q[\phi(x_1, 0) + c(x_1)]$, so that $\phi(x, 0) = -c(x)$, $\forall x \in X$, which is (2.8) for $p = 0$. Finally, (2.4) follows from (2.3), (2.8) for all $p \in [0, 1]$, (3.2), (3.3), (2.11), and (1.5).

LEMMA 3.2. *If I_n $(n = 2, 3, \ldots)$ is generalized-recursive and has the* (2.3) *sum form on $X_n \times \Gamma_n$, with ϕ, ϕ_0, g given by* (2.9) *and* (2.11), *then I_n has either form* (2.5) *for some map $a: X \to R$ and constant α $(\neq 0, 1)$, or form* (2.7) *for some maps f, h: $X \to R$ with $h(0) = 0$.*

PROOF. Substituting (2.9) and (2.11) into (2.12), we get

$$\begin{aligned}&\left[c(x_1 \cup x_2) - c(x_1) - c(x_2)\right]\left[1 - (p+q)^{\alpha}\right]\\&\quad = \left[b(x_1 \cup x_2)(p+q) - b(x_1)p - b(x_2)q\right]\left[1 - (p+q)^{\alpha-1}\right] \qquad (3.4)\end{aligned}$$

for all $(p, q) \in D'$ and all $(x_1, x_2) \in X_2$. We consider two cases.

Case 1. Suppose $\alpha \neq 0$. We again get (3.2) upon equating constants (in p and q). Then (3.4) yields (since $\alpha \neq 1$)

$$b(x_1 \cup x_2)(p+q) = b(x_1)p + b(x_2)q.$$

Equating coefficients of p, we obtain $b(x_1 \cup x_2) = b(x_1)$, so there is a constant γ such that

$$b(x) = \gamma, \qquad \forall x \in X. \qquad (3.5)$$

By (2.9), (3.2), and (3.5), (2.12) with $p = 0 \neq q$ becomes $\phi(x_1, 0) + c(x_1) = q^{\alpha}[\phi(x_1, 0) + c(x_1)]$. Thus we have (2.9) for $p = 0$, i.e., $\phi(x, 0) = -c(x)$, $\forall x \in X$. Now (2.5) follows from (2.3), (2.11), (2.9) for all $p \in [0, 1]$, (3.2), and (3.5).

Case 2. Suppose $\alpha = 0$. Then (3.4) becomes, after multiplication by $(p + q)$,

$$0 = \left[b(x_1 \cup x_2)(p+q) - b(x_1)p - b(x_2)q\right](p + q - 1).$$

Examining coefficients of p^2, we see that $b(x_1 \cup x_2) = b(x_1)$, so that (3.5) again holds for some constant γ.

However, (2.9) does not hold for $p = 0$. Indeed, $\phi(x, 0)$ is arbitrary, say

$$h(x) := \phi(x, 0), \qquad \forall x \in X. \qquad (3.6)$$

Also, (3.2) does not hold, so we define $f: X \to R$ by

$$f(x) := a(x) - c(x), \qquad \forall x \in X. \qquad (3.7)$$

Finally, (2.7) follows from (2.3), (2.11), (2.9) for $\alpha = 0$ and $p \in]0, 1]$, (3.5), (3.6), and (3.7). Moreover, $h(0) = 0$ by (3.6) and (2.3A).

LEMMA 3.3. *If I_n $(n = 2, 3, \ldots)$ is generalized-recursive and has the* (2.3) *sum form on $X_n \times \Gamma_n$, with ϕ, ϕ_0, g given by* (2.10) *and (provided $g \not\equiv 0$)* (2.11), *then I_n has one of the forms* (2.4), (2.6), (2.7).

PROOF. We consider three cases.

Case 1. Suppose $g \not\equiv 0$ and $g \not\equiv 1$. By (1.4), this means that g is nonconstant. By (2.10) and (2.11), (2.12) becomes

$$\begin{aligned}&\left[c(x_1 \cup x_2) - c(x_1) - c(x_2)\right]\left[1 - g(p+q)\right]\\&\quad = \left[a(x_1 \cup x_2)(p+q) - a(x_1)p - a(x_2)q\right]\left[1 - g(p+q)(p+q)^{-1}\right] \qquad (3.8)\end{aligned}$$

for all $(p, q) \in D'$ and for all $(x_1, x_2) \in X_2$. Since g is nonconstant, we again have (3.2) by equating constants. Moreover, if $g(p) \equiv p$, then I_n has the form (2.4) by Lemma 3.1. Otherwise, (3.8) and (3.2) give $a(x_1 \cup x_2)(p + q) = a(x_1)p + a(x_2)q$, which again means that we have

$$a(x) = \gamma \text{ (constant)}, \qquad \forall x \in X. \tag{3.9}$$

Finally, we again get $\phi(x, 0) = -c(x)$ (i.e., (2.10) for $p = 0$) by (2.12) with $p = 0 \neq q$, (2.10), (2.11), (3.9), (3.2), and the nonconstance of g. By the additional use of (2.3), we find that $I_n \equiv 0$, a special case of (2.4), (2.6), or (2.7).

Case 2. Suppose $g \equiv 1$. By (2.10) and (2.11), (2.12) becomes, after multiplication by $(p + q)$,

$$[a(x_1)p + a(x_2)q - a(x_1 \cup x_2)(p + q)](p + q - 1) = 0,$$

for $(p, q) \in D'$, $(x_1, x_2) \in X_2$. Upon examining the coefficient of p^2, we find that (3.9) holds once again.

But, as in Case 2 of the proof of Lemma 3.2, $\phi(x, 0)$ is arbitrary, and we define $h: X \to R$ by (3.6). By (2.3), (2.11), (2.10) for $p \in]0, 1]$, (3.9), and (3.6), we obtain (2.7) with map $f := -c$.

Case 3. Suppose $g \equiv 0$. By (2.10), (2.12) becomes

$$a(x_1)p + a(x_2)q - a(x_1 \cup x_2)(p + q) = c(x_1) + c(x_2) - c(x_1 \cup x_2),$$

for $(p, q) \in D'$, $(x_1, x_2) \in X_2$. Again, (3.2) and (3.9) follow.

Next, (2.12) with $p \neq 0 = q$ gives $\phi(x_2, 0) = -c(x_2)$, so (2.10) holds for all $p \in [0, 1]$. Now, by (2.3), (2.10), (3.2), and (3.9), we get form (2.6) for I_n, where $f(x) := -c(x) + \phi_0(x, 1)$.

4. Remarks. There is an analytic connection between (2.4) and (2.5). If $b: X \to R$ is defined by

$$b(x) := a(x) - \gamma(2^{1-\alpha} - 1)^{-1}, \qquad \forall x \in X,$$

then (2.5) goes over into

$$I_n\begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = \gamma H_n^\alpha(p_1, p_2, \ldots, p_n) + \sum_{i=1}^n p_i^\alpha b(x_i) - b\left(\bigcup_{i=1}^n x_i\right), \tag{4.1}$$

where $H_n^\alpha: \Gamma_n \to R$ is the entropy of degree α, defined in (1.6). As α tends to 1, the entropy in (4.1) tends to (2.4).

Finally, the results of Theorem 2.2 can be extended, with modifications, to the domain $X^n \times [0, 1]^n$. The representations on this larger domain should, in general, contain the term $\phi_0(\bigcup_{i=1}^n x_i, p(\bigcup_{i=1}^n x_i))$ and not make use of (2.11).

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