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REAL MINIMAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A characterization of the geodesic minimal hypersphere in a complex projective space is given.

Introduction. Let CP^n denote a complex n-dimensional projective space equipped with the Fubini-Study metric normalized so that the maximum sectional curvature is 4. We consider the Hopf fibration $\bar{\pi}$:

$$S^1 \to S^{2n+1} \xrightarrow{\bar{\pi}} CP^n,$$

where S^k denotes the Euclidean sphere of curvature 1. In S^{2n+1} we have the family of generalized Clifford surfaces whose fibres lie in complex subspaces (see [1]):

$$M_{2p+1,2q+1} = S^{2p+1}\left(\sqrt{\frac{2p+1}{2n}}\right) \times S^{2q+1}\left(\sqrt{\frac{2q+1}{2n}}\right),$$

where $p + q = n - 1$. Then we have a fibration π:

$$S^1 \to M_{2p+1,2q+1} \xrightarrow{\pi} M^C_{p,q},$$

compatible with $\bar{\pi}$. In the special case $p = 0$, $M^C_{0,n-1}$ is called the geodesic minimal hypersphere (see [5]).

In this paper we shall prove the following result.

THEOREM. *Let M be a compact orientable real minimal hypersurface of CP^n. If the sectional curvature K of M satisfies $K \geqslant 1/(2n-1)$, then M is the geodesic minimal hypersphere $M^C_{0,n-1}$.*

1. Auxiliary results. Let M be a real hypersurface isometrically immersed in CP^n. We denote by J the almost complex structure of CP^n and by C a unit normal of M in CP^n. For any vector field X tangent to M we put

$$JX = PX + f(X)C, \tag{1.1}$$

where PX is the tangential part of JX and f is a 1-form. Then P is an endomorphism on the tangent bundle of M. We now put $U = -JC$. Then U is the unit vector field tangent to M. From (1.1) we have

$$P^2X = -X + f(X)U, \qquad PU = 0. \tag{1.2}$$

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The Riemannian metric tensor field of M will be denoted by g. Then we obtain

$$g(PX, Y) + g(X, PY) = 0, \qquad f(X) = g(X, U). \tag{1.3}$$

We denote by $\overline{\nabla}$ (resp. ∇) the operator of covariant differentiation of CP^n (resp. ∇). Then the Gauss and Weingarten formlas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)C \quad \text{and} \quad \overline{\nabla}_X C = -AX$$

for any vector fields X and Y tangent to M. We call A the second fundamental form of M, which can be considered as a symmetric $(2n-1, 2n-1)$-matrix.

From (1.1) and Gauss and Weingarten formulas we have

$$\nabla_X U = PAX, \tag{1.4}$$

$$(\nabla_X P)Y = f(Y)AX - g(AX, Y)U. \tag{1.5}$$

Let R denote the Riemannian curvature tensor of M. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX - g(PX, Z)PY \\ +2g(X, PY)PZ + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = f(X)PY - f(Y)PX + 2g(X, PY)U. \tag{1.7}$$

From (1.6) the Ricci tensor S of M is given by

$$S(X, Y) = 2(n-1) + 3g(PX, PY) + \operatorname{Tr} A g(AX, Y) - g(A^2X, Y), \tag{1.8}$$

where $\operatorname{Tr} A$ denotes the trace of A. If $\operatorname{Tr} A = 0$, then M is said to be minimal.

LEMMA 1. *Let M be a real hypersurface of CP^n. If M is minimal, then*

$$\operatorname{div}(\nabla_U U) = 2(n-1) - \operatorname{Tr} A^2 + \tfrac{1}{2}|[P, A]|^2.$$

PROOF. First of all, we have **[6]**

$$\operatorname{div}(\nabla_U U) - \operatorname{div}((\operatorname{div} U)U) = S(U, U) + \tfrac{1}{2}|L(U)g|^2 - |\nabla U|^2 - (\operatorname{div} U)^2,$$

where $L(U)g$ denotes the Lie derivative of g with respect to U and $|\ |$ denotes the length with respect to g. Since P is skew-symmetric and A is symmetric, (1.4) implies that $\operatorname{div} U = 0$ and hence $\operatorname{div}((\operatorname{div} U)U) = 0$. Thus

$$\operatorname{div}(\nabla_U U) = S(U, U) + \tfrac{1}{2}|L(U)g|^2 - |\nabla U|^2. \tag{1.9}$$

On the other hand, from (1.2) and (1.3), we find

$$|\nabla U|^2 = \operatorname{Tr} A^2 - g(A^2U, U).$$

From the minimality of M and (1.8) we have

$$S(U, U) = 2(n-1) - g(A^2U, U).$$

Substituting these equations into (1.9), we have

$$\operatorname{div}(\nabla_U U) = 2(n-1) - \operatorname{Tr} A^2 + \tfrac{1}{2}|L(U)g|^2. \tag{1.10}$$

From (1.4) we see that

$$(L(U)g)(X, Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X) = g((PA - AP)X, Y),$$

from which $|L(U)g|^2 = |[P, A]|^2$, where $[P, A] = PA - AP$. From this and (1.10) we have our equation.

In the sequel, we compute the Laplacian for the second fundamental form A of M (see [4]).

LEMMA 2. *Let M be a real minimal hypersurface of CP^n. Then*

$$g(\nabla^2 A, A) = \sum_{i,j} g\big((R(e_j, e_i)A)e_j, Ae_i\big) - 3\mathrm{Tr}\, A^2 + \frac{3}{2}|[P, A]|^2,$$

where $\{e_i\}$ denotes an orthonormal frame for M.

PROOF. Let X be an arbitrary vector tangent to M. Then (1.7) implies

$$\sum_j (\nabla_{e_j} A)e_j = 0. \tag{1.11}$$

From (1.4), (1.5), (1.7) and (1.11) we obtain

$$\begin{aligned} g(\nabla^2 A, A) &= \sum_{i,j} g\big((\nabla_{e_j} \nabla_{e_j} A)e_i, Ae_i\big) \\ &= \sum_{i,j} g\big((R(e_j, e_i)A)e_j, Ae_i\big) - 3g(A^2U, U) + 3\mathrm{Tr}(AP)^2. \end{aligned} \tag{1.12}$$

Since $\mathrm{Tr}(AP)^2 = \mathrm{Tr}\, A^2P^2 + \frac{1}{2}|[P, A]|^2$, we obtain

$$-3g(A^2U, U) + 3\mathrm{Tr}(AP)^2 = -3\mathrm{Tr}\, A^2 + \frac{3}{2}|[P, A]|^2. \tag{1.13}$$

Substituting (1.13) into (1.12), we have our assertion.

We use the following

LEMMA 3 ([2]). *Let M be a real hypersurface of CP^n. Then we have*

$$|\nabla A|^2 \geqslant 4(n - 1).$$

2. Proof of theorem. Since M is compact orientable, Lemmas 1, 2 and 3 imply

$$\begin{aligned} 0 &\leqslant \int_M \Big[|\nabla A|^2 - 4(n - 1) + \frac{1}{2}|[P, A]|^2\Big] {*1} \\ &= \int_M \Big[\mathrm{Tr}\, A^2 - \sum_{i,j} g\big((R(e_j, e_i)A)e_j, Ae_i\big)\Big] {*1}, \end{aligned} \tag{2.1}$$

where $*1$ denotes the volume element of M. We choose an orthonormal frame $\{e_i\}$ of M such that $Ae_i = \lambda_i e_i$ ($i = 1, \ldots, 2n - 1$). Then

$$\begin{aligned} -\sum_{i,j} g\big((R(e_j, e_i)A)e_j, Ae_i\big) &= \sum_{i,j} \big[g\big(AR(e_j, e_i)e_j, Ae_i\big) - g\big(R(e_j, e_i)Ae_j, Ae_i\big)\big] \\ &= -\frac{1}{2} \sum_{i,j} (\lambda_j - \lambda_i)^2 K_{ij}, \end{aligned}$$

where K_{ij} denotes the sectional curvature of M spanned by e_i and e_j. In accordance with the assumption, we get $K_{ij} \geqslant 1/(2n - 1)$ and hence

$$-\sum_{i,j} g\big((R(e_j, e_i)A)e_j, Ae_i\big) \leqslant -\frac{1}{2(2n - 1)} \sum_{i,j} (\lambda_j - \lambda_i)^2 = -\mathrm{Tr}\, A^2.$$

Therefore, the right-hand side of (2.1) is nonpositive. Hence we have $|\nabla A|^2 = 4(n - 1)$ and $PA = AP$. Applying a theorem of [2] or [3], we see that M is $M^C_{p,q}$. On the other hand, if $p, q \geqslant 1$, then the sectional curvature K of $M_{p,q}^C$ takes value 0 for some plane section. But the sectional curvature K of $M_{0,n-1}^C$ satisfies $K \geqslant 1/(2n-1)$. Consequently, M is the geodesic minimal hypersphere $M_{0,n-1}^C$.

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