REAL MINIMAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A characterization of the geodesic minimal hypersphere in a complex projective space is given.

Introduction. Let \( CP^n \) denote a complex \( n \)-dimensional projective space equipped with the Fubini-Study metric normalized so that the maximum sectional curvature is \( 4 \). We consider the Hopf fibration

\[
S^1 \rightarrow S^{2n+1} \rightarrow CP^n,
\]

where \( S^k \) denotes the Euclidean sphere of curvature 1. In \( S^{2n+1} \) we have the family of generalized Clifford surfaces whose fibres lie in complex subspaces (see [1]):

\[
M_{2p+1,2q+1} = S^{2p+1}\left(\sqrt{\frac{2p+1}{2n}}\right) \times S^{2q+1}\left(\sqrt{\frac{2q+1}{2n}}\right),
\]

where \( p + q = n - 1 \). Then we have a fibration \( \pi \):

\[
S^1 \rightarrow M_{2p+1,2q+1} \rightarrow M_{p,q}^C,
\]

compatible with \( \bar{\pi} \). In the special case \( p = 0 \), \( M_{0,n-1}^C \) is called the geodesic minimal hypersphere (see [5]).

In this paper we shall prove the following result.

**Theorem.** Let \( M \) be a compact orientable real minimal hypersurface of \( CP^n \). If the sectional curvature \( K \) of \( M \) satisfies \( K \geq 1/(2n-1) \), then \( M \) is the geodesic minimal hypersphere \( M_{0,n-1}^C \).

1. Auxiliary results. Let \( M \) be a real hypersurface isometrically immersed in \( CP^n \). We denote by \( J \) the almost complex structure of \( CP^n \) and by \( C \) a unit normal of \( M \) in \( CP^n \). For any vector field \( X \) tangent to \( M \) we put

\[
JX = PX + f(X)C,
\]

where \( PX \) is the tangential part of \( JX \) and \( f \) is a 1-form. Then \( P \) is an endomorphism on the tangent bundle of \( M \). We now put \( U = -JC \). Then \( U \) is the unit vector field tangent to \( M \). From (1.1) we have

\[
P^2X = -X + f(X)U, \quad PU = 0.
\]
The Riemannian metric tensor field of $M$ will be denoted by $g$. Then we obtain
\[ g(\mathbf{PX}, \mathbf{Y}) + g(\mathbf{X}, \mathbf{PY}) = 0, \quad f(\mathbf{X}) = g(\mathbf{X}, \mathbf{U}). \quad (1.3) \]

We denote by $\nabla$ (resp. $\nabla$) the operator of covariant differentiation of $\mathbb{C}P^n$ (resp. $\nabla$). Then the Gauss and Weingarten formulas are given respectively by
\[
\nabla_X Y = \nabla_X Y + g(AX, Y)C \quad \text{and} \quad \nabla_X C = -AX
\]
for any vector fields $X$ and $Y$ tangent to $M$. We call $A$ the second fundamental form of $M$, which can be considered as a symmetric $(2n-1, 2n-1)$-matrix.

From (1.1) and Gauss and Weingarten formulas we have
\[
\nabla_X U = PAX, \quad (1.4)
\]
\[
(\nabla_X P) Y = f(Y)AX - g(AX, Y)U. \quad (1.5)
\]

Let $R$ denote the Riemannian curvature tensor of $M$. Then we have the following Gauss and Codazzi equations:
\[
R(\mathbf{X}, \mathbf{Y})\mathbf{Z} = g(\mathbf{Y}, \mathbf{Z})\mathbf{X} - g(\mathbf{X}, \mathbf{Z})\mathbf{Y} + g(\mathbf{PY}, \mathbf{Z})\mathbf{PX} - g(\mathbf{PX}, \mathbf{Z})\mathbf{PY} + 2g(\mathbf{X}, \mathbf{PY})\mathbf{PZ} + g(A\mathbf{Y}, \mathbf{Z})A\mathbf{X} - g(A\mathbf{X}, \mathbf{Z})A\mathbf{Y}, \quad (1.6)
\]
\[
(\nabla_X A) Y - (\nabla_Y A) X = f(X)PY - f(Y)PX + 2g(\mathbf{X}, \mathbf{PY})\mathbf{U}. \quad (1.7)
\]

From (1.6) the Ricci tensor $S$ of $M$ is given by
\[
S(\mathbf{X}, \mathbf{Y}) = 2(n-1) + 3g(\mathbf{PX}, \mathbf{PY}) + Tr Ag(\mathbf{AX}, \mathbf{Y}) - g(A^2\mathbf{X}, \mathbf{Y}), \quad (1.8)
\]
where $Tr A$ denotes the trace of $A$. If $Tr A = 0$, then $M$ is said to be minimal.

**Lemma 1.** Let $M$ be a real hypersurface of $\mathbb{C}P^n$. If $M$ is minimal, then
\[
\text{div}(\nabla_U U) = 2(n-1) - Tr A^2 + \frac{1}{2}|[P, A]|^2. \quad (1.10)
\]

**Proof.** First of all, we have [6]
\[
\text{div}(\nabla_U U) - \text{div}(\text{div}(U) U) = S(U, U) + \frac{1}{2}|L(U)g|^2 - |\nabla U|^2 - (\text{div} U)^2,
\]
where $L(U)g$ denotes the Lie derivative of $g$ with respect to $U$ and $||$ denotes the length with respect to $g$. Since $P$ is skew-symmetric and $A$ is symmetric, (1.4) implies that $\text{div} U = 0$ and hence $\text{div}(\text{div}(U) U) = 0$. Thus
\[
\text{div}(\nabla_U U) = S(U, U) + \frac{1}{2}|L(U)g|^2 - |\nabla U|^2. \quad (1.9)
\]
On the other hand, from (1.2) and (1.3), we find
\[
|\nabla U|^2 = Tr A^2 - g(A^2 U, U).
\]
From the minimality of $M$ and (1.8) we have
\[
S(U, U) = 2(n-1) - g(A^2 U, U).
\]
Substituting these equations into (1.9), we have
\[
\text{div}(\nabla_U U) = 2(n-1) - Tr A^2 + \frac{1}{2}|L(U)g|^2. \quad (1.10)
\]
From (1.4) we see that
\[
(L(U)g)(X, Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X) = g((PA - AP)X, Y),
\]
from which $|L(U)g|^2 = |[P, A]|^2$, where $[P, A] = PA - AP$. From this and (1.10) we have our equation.
In the sequel, we compute the Laplacian for the second fundamental form $A$ of $M$ (see [4]).

**Lemma 2.** Let $M$ be a real minimal hypersurface of $\mathbb{C}P^n$. Then

$$g(\nabla^2 A, A) = \sum_{ij} g((R(e_j, e_i)A)e_j, Ae_i) - 3\text{Tr} A^2 + \frac{3}{2} ||[P, A]||^2,$$

where $\{e_i\}$ denotes an orthonormal frame for $M$.

**Proof.** Let $X$ be an arbitrary vector tangent to $M$. Then (1.7) implies

$$\nabla^2 e_i = 0. \quad (1.11)$$

From (1.4), (1.5), (1.7) and (1.11) we obtain

$$g(\nabla^2 A, A) = \sum_{ij} \left( g(\nabla_{e_i} \nabla_{e_j} A) e_i, Ae_i \right)$$

$$= \sum_{ij} g((R(e_j, e_i)A)e_j, Ae_i) - 3g(A^2 U, U) + 3\text{Tr}(AP)^2. \quad (1.12)$$

Since $\text{Tr}(AP)^2 = \text{Tr} A^2 P^2 + \frac{1}{2} ||[P, A]||^2$, we obtain

$$-3g(A^2 U, U) + 3\text{Tr}(AP)^2 = -3\text{Tr} A^2 + \frac{3}{2} ||[P, A]||^2. \quad (1.13)$$

Substituting (1.13) into (1.12), we have our assertion.

We use the following

**Lemma 3 ([2]).** Let $M$ be a real hypersurface of $\mathbb{C}P^n$. Then we have

$$|\nabla A|^2 > 4(n - 1).$$

2. **Proof of theorem.** Since $M$ is compact orientable, Lemmas 1, 2 and 3 imply

$$0 < \int_M \left[ |\nabla A|^2 - 4(n - 1) + \frac{1}{2} ||[P, A]||^2 \right] \cdot 1$$

$$= \int_M \left[ \text{Tr} A^2 - \sum_{ij} g((R(e_j, e_i)A)e_j, Ae_i) \right] \cdot 1, \quad (2.1)$$

where $\cdot 1$ denotes the volume element of $M$. We choose an orthonormal frame $\{e_i\}$ of $M$ such that $Ae_i = \lambda_i e_i$ ($i = 1, \ldots, 2n - 1$). Then

$$-\sum_{ij} g((R(e_j, e_i)A)e_j, Ae_i) = \sum_{ij} \left[ g(AR(e_j, e_i)e_j, Ae_i) - g(R(e_j, e_i)e_j, Ae_i) \right]$$

$$= -\frac{1}{2} \sum_{ij} (\lambda_i - \lambda_j)^2 K_{ij},$$

where $K_{ij}$ denotes the sectional curvature of $M$ spanned by $e_i$ and $e_j$. In accordance with the assumption, we get $K_{ij} > 1/(2n - 1)$ and hence

$$-\sum_{ij} g((R(e_j, e_i)A)e_j, Ae_i) < -\frac{1}{2(2n - 1)} \sum_{ij} (\lambda_i - \lambda_j)^2 = -\text{Tr} A^2.$$

Therefore, the right-hand side of (2.1) is nonpositive. Hence we have $|\nabla A|^2 = 4(n - 1)$ and $PA = AP$. Applying a theorem of [2] or [3], we see that $M$ is $M_{p,q}^C$. On
the other hand, if $p, q > 1$, then the sectional curvature $K$ of $M^{p,q}_p$ takes value 0 for some plane section. But the sectional curvature $K$ of $M^n_{0,n-1}$ satisfies $K > 1/(2n - 1)$. Consequently, $M$ is the geodesic minimal hypersphere $M^n_{0,n-1}$.

REFERENCES


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