THE HANF NUMBER OF $L_{\omega_1^{\omega_1}}$

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ABSTRACT. A model of set theory is constructed in which the Hanf number of $L_{\omega_1^{\omega_1}}$ is below the first weakly compact cardinal. This answers a question of J. Silver.

1. Introduction. J. Silver proved in [9] that if the smallest Erdös cardinal $\kappa(\omega)$ exists, then $h_{\omega_1^{\omega_1}}$, the Hanf number of $L_{\omega_1^{\omega_1}}$, exceeds $\kappa(\omega)$, and therefore also the first weakly compact cardinal. He left it as an open problem whether $h_{\omega_1^{\omega_1}}$ could, in the absence of $\kappa(\omega)$, be below the first weakly compact cardinal. The purpose of this paper is to give an affirmative answer to this problem.

K. Kunen proved in [5] that $h_{\omega_1^{\omega_1}}$ can be even larger than a measurable cardinal (indeed this always takes place in the inner model $L^*$), but for trivial reasons $h_{\omega_1^{\omega_1}}$ is below the first strongly compact cardinal. M. Magidor showed in [7] that the first measurable cardinal could be strongly compact, whence $h_{\omega_1^{\omega_1}}$ could be below the first measurable cardinal. We prove this result without assuming the consistency of a strongly compact cardinal. Likewise our method yields a model in which $h_{\omega_1^{\omega_1}}$ is below the first Ramsey cardinal.

2. Preliminaries. The infinitary language $L_{\omega_1^{\omega_1}}$ extends the usual first order logic by allowing countable disjunctions and conjunctions

$$\phi_0 \lor \phi_1 \lor \ldots \lor \phi_n \lor \ldots, \quad \phi_0 \land \phi_1 \land \ldots \land \phi_n \land \ldots,$$

and quantification over countably many variables

$$\exists x_0 \ldots x_n \ldots \phi(x_0, \ldots, x_n, \ldots), \quad \forall x_0 \ldots x_n \ldots \phi(x_0, \ldots, x_n, \ldots).$$

For a rigorous definition of $L_{\omega_1^{\omega_1}}$ see [1].

Hanf numbers are related to spectra. If $\phi \in L_{\omega_1^{\omega_1}}$, the spectrum of $\phi$ is the class

$$\text{Sp}(\phi) = \{|A| \mid A \models \phi\}.$$

If $\text{Sp}(\phi)$ is not a proper class, we can define

$$\sup(\phi) = \sup \text{Sp}(\phi).$$

Otherwise we define $\sup(\phi) = 0$. The Hanf number of $L_{\omega_1^{\omega_1}}$ is

$$h_{\omega_1^{\omega_1}} = \sup \{ \sup(\phi) \mid \phi \in L_{\omega_1^{\omega_1}} \}.$$
Lemma 1. \( h_{\omega_1} \) exceeds the first inaccessible (hyperinaccessible, Mahlo, hyper-Mahlo) cardinal (if any).

J. Silver proved that the predicate \( \kappa \rightarrow \langle \omega \rangle^{<\kappa} \) is \( \Pi^1 \) thereby extending the proof of Lemma 1 to a proof of (see [9])

Lemma 2. If \( \kappa(\omega) \) exists, then \( h_{\omega_1} > \kappa(\omega) \). Moreover, if \( \kappa(\alpha) \) exists for all \( \alpha < \omega_1 \), then \( h_{\omega_1} > \sup\{ \kappa(\omega)|\alpha < \omega_1 \} \).

Similarly, \( h_{\omega_1} \) exceeds the first subtle cardinal (if any).

The set-theoretical notation of the next two sections follows that of [3].

3. The main result.

Theorem 1. If the theory

\[
\text{ZFC} + \text{"there is a proper class of weakly compact cardinals"}
\]

is consistent, then so is the theory

\[
\text{ZFC} + \text{"} h_{\omega_1} < \text{the first weakly compact cardinal"}.
\]

Proof. Let us assume that \( V = L \) and that \( \rho_\alpha (\alpha \in On) \) is the proper class of all weakly compact cardinals in ascending order. We may assume that \( \sup\{ \rho_\alpha|\alpha < \eta \} \) is singular for every limit \( \eta \) (otherwise we move into a submodel of the universe).

Let \( P \) be the notion of forcing which results if we add, via backward Easton, a Cohen-subset for every inaccessible cardinal. Thus \( P \) is the direct limit of a system \( \{ P_\alpha|\alpha \in On \} \) of notions of forcing defined as follows: For inaccessible \( \nu \), \( P_\alpha \) is the direct limit of \( \{ P_\alpha|\alpha < \nu \} \), and for other limit \( \nu \) the corresponding inverse limit. At successor stages we let \( P_{\alpha+1} = P_\alpha * C \), where for inaccessible \( \alpha \), \( C \) is the notion of forcing for adding a Cohen subset for \( \alpha \), and for other \( \alpha \), \( C = \{ \emptyset \} \). For details concerning backward Easton forcing the reader is referred to [3], [4] and [8].

Following the idea of the proof of Theorem 88 in [3] (or §25 in [4]) one can prove that \( P \) preserves weak compactness. On the other hand, no new cardinals become weakly compact, because weak compactness relativizes to \( L \).

K. Kunen describes in [6] a notion of forcing \( R \) which can be used to add a \( \kappa \)-Souslin tree \( T \) for a regular uncountable cardinal \( \kappa \). This notion of forcing has the additional property that if \( T \) is later killed by forcing with its reverse ordering \( T \) the resulting product forcing \( R * T \) is isomorphic to the notion of forcing used for adding a Cohen subset for \( \kappa \). This fact has a basic role in our proof.

We shall now extend \( P \) by a new backward Easton construction. For this end, let \( Q_0 = P \). For limit \( \nu \), let \( Q_\nu \) be the inverse limit of \( \{ Q_\alpha|\alpha < \nu \} \). At successor stages we let \( Q_{\alpha+1} = Q_\alpha * R_\alpha \), where \( R_\alpha \) is the notion of forcing, cited above, for adding a \( \rho_\alpha \)-Souslin tree \( T_\alpha \) for \( \rho_\alpha \). Finally, let \( Q \) be the direct limit of \( \{ Q_\alpha|\alpha \in On \} \). Note that forcing with \( Q_\alpha \) kills the weak compactness of \( \rho_\beta (\beta < \alpha) \) because of the tree \( T_\beta \). On the other hand, \( Q_\alpha \) is mild for \( \rho_\alpha \) (as we have no regular limits of weakly compact cardinals) and therefore preserves \( \rho_\alpha \) weakly compact, making it in fact the first weakly compact cardinal.
Our third backward Easton construction kills the trees $T_\alpha$ ($\alpha \in \text{On}$) one by one. Let $U_0 = Q$. For limit $\nu$, let $U_\nu$ be the inverse limit of $\{U_\alpha | \alpha < \nu\}$. At successor stages we let $U_{\alpha+1} = U_\alpha * T_\alpha$, where $T_\alpha$ is the reverse ordering of $T_\alpha$. Finally let $U$ be the direct limit of $\{U_\alpha | \alpha \in \text{On}\}$.

Using the facts that (1) $U_\alpha$ is isomorphic to the backward Easton extension of $P$ which adds at first $R_\beta$ and $T_\beta$ for $\beta < \alpha$ and then $R_\beta$ for $\beta > \alpha$, (2) $R_\beta * T_\beta$ is isomorphic to the set of Cohen conditions for $\rho_\beta$, and (3) adding two Cohen subsets is (up to $\equiv$) the same as adding just one Cohen subset, one can construct an isomorphism $i: U \cong P$.

We claim that for some $\alpha$ and for some $p \in Q_\alpha$ we have

$$p \models_{Q_\alpha} \omega_1 < \rho_\alpha.$$ 

This will prove the theorem, as $\rho_\alpha$ is the smallest weakly compact cardinal in any extension by $Q_\alpha$.

Suppose the claim fails. Then for every $\alpha$ there is a $p_\alpha \in Q_\alpha$ such that

$$p_\alpha \not\models Q_\alpha \exists \phi_\alpha \in L_{\omega_1, \omega_1}(\check{\rho}_\alpha < \sup(\phi_\alpha)).$$

Note that for all notions of forcing used are $< \rho_0$-distributive. Therefore we may pick a $\phi_\alpha \in L_{\omega_1, \omega_1}$ such that

$$p_\alpha \models Q_\alpha \check{\rho}_\alpha < \sup(\phi_\alpha). \quad (\ast)$$

By a cardinality argument there is a single sentence $\phi_\alpha \in L_{\omega_1, \omega_1}$ such that for arbitrarily large $\beta$

$$p_\beta \models Q_\beta \check{\rho}_\beta < \sup(\phi_\alpha).$$

Using the preservation of the satisfaction relation of $L_{\omega_1, \omega_1}$ under extensions not adding new countable sets of ordinals, we can find for arbitrarily large $\beta$ a condition $q_\beta \in U$ such that

$$q_\beta \models_{U} \text{Sp}(\check{\phi}_\alpha) - \check{\rho}_\beta \neq \emptyset.$$ 

By the isomorphism of $U$ and $P$, and by the homogeneity of $P$ (see [8]) we obtain

$$1 \models P \text{Sp}(\check{\phi}_\alpha) - \check{\rho}_\beta \neq \emptyset$$

for arbitrarily large $\beta$ (1 is the maximal element of $P$). Therefore $1 \models P \sup(\check{\phi}_\alpha) = 0$, and by another absoluteness argument,

$$1 \models Q_\alpha \sup(\check{\phi}_\alpha) = 0.$$ 

This contradicts (\ast), and therefore establishes the claim. This ends the proof of Theorem 1.

4. Generalizations and related results. Examination of the proof of Theorem 1 reveals that $L_{\omega_1, \omega_1}$ can be considerably extended without affecting the result. The proof works equally well for example for $L_{\omega_1, \omega_2}$. More interestingly, we can add new quantifiers to $L_{\omega_1, \omega_1}$. A simple quantifier not definable even in $L_{\omega_1, \omega_1}$ is the Härting-quantifier

$$IxyA(x)B(y) \text{ if and only if } |A(\cdot)| = |B(\cdot)|.$$
Theorem 1 remains true if this quantifier be added to $L_{\omega_1^{\omega_1}}$. This is noteworthy because the Hanf number of the extension of $L_{\omega_1^{\omega_1}}$ by $I$ can be even bigger than a supercompact cardinal (a proof of this will appear elsewhere).

It is apparent that the proof of Theorem 1 extends immediately to a proof of

**Theorem 2.** If the theory

$$ZFC + \text{"there is a proper class of weakly compact cardinals"}$$

is consistent, then so is the theory

$$ZFC + \text{the sentence "the first weakly compact cardinal exceeds the Hanf number of the extension of } L_{\omega_1^{\omega_1}} \text{ by } Q\text{" for every generalized quantifier } Q \text{ the definition of which is provably absolute with respect to extensions which preserve cardinals and cofinalities.}$$

The role played by weakly compact cardinals in Theorem 1 is by no means unique. For example, only slight changes are needed to prove

**Theorem 3.** If the theory

$$ZFC + \text{"there is a proper class of measurable cardinals"}$$

is consistent, then so is the theory

$$ZFC + \ "h_{\omega_1^{\omega_1}} < \text{the first measurable cardinal}".$$  

T. Jech has pointed out that Ramsey cardinals are also preserved under forcing with $P$. Thus Theorem 1 holds if “weakly compact” is replaced by “Ramsey”. This answers problem 9 of H. Friedman [2].

**References**


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