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# THE HANF NUMBER OF $L_{\omega_1\omega_1}$

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ABSTRACT. A model of set theory is constructed in which the Hanf number of $L_{\omega_1\omega_1}$ is below the first weakly compact cardinal. This answers a question of J. Silver.

**1. Introduction.** J. Silver proved in [9] that if the smallest Erdös cardinal $\kappa(\omega)$ exists, then $h_{\omega_1\omega_1}$, the Hanf number of $L_{\omega_1\omega_1}$, exceeds $\kappa(\omega)$, and therefore also the first weakly compact cardinal. He left it as an open problem whether $h_{\omega_1\omega_1}$ could, in the absence of $\kappa(\omega)$, be below the first weakly compact cardinal. The purpose of this paper is to give an affirmative answer to this problem.

K. Kunen proved in [5] that $h_{\omega_1\omega_1}$ can be even larger than a measurable cardinal (indeed this always takes place in the inner model $L^{\mu}$), but for trivial reasons $h_{\omega_1\omega_1}$ is below the first strongly compact cardinal. M. Magidor showed in [7] that the first measurable cardinal could be strongly compact, whence $h_{\omega_1\omega_1}$ could be below the first measurable cardinal. We prove this result without assuming the consistency of a strongly compact cardinal. Likewise our method yields a model in which $h_{\omega_1\omega_1}$ is below the first Ramsey cardinal.

**2. Preliminaries.** The infinitary language $L_{\omega_1\omega_1}$ extends the usual first order logic by allowing countable disjunctions and conjunctions

$$\phi_0 \vee \phi_1 \vee \cdots \vee \phi_n \vee \cdots, \qquad \phi_0 \wedge \phi_1 \wedge \cdots \wedge \phi_n \wedge \cdots,$$

and quantification over countably many variables

$$\exists x_0 \ldots x_n \ldots \phi(x_0, \ldots, x_n, \ldots), \qquad \forall x_0 \ldots x_n \ldots \phi(x_0, \ldots, x_n, \ldots).$$

For a rigorous definition of $L_{\omega_1\omega_1}$ see [1].

Hanf numbers are related to *spectra*. If $\phi \in L_{\omega_1\omega_1}$, the *spectrum* of $\phi$ is the class

$$\mathrm{Sp}(\phi) = \{|A| \mid A \models \phi\}.$$

If $\mathrm{Sp}(\phi)$ is not a proper class, we can define

$$\sup(\phi) = \sup \mathrm{Sp}(\phi).$$

Otherwise we define $\sup(\phi) = 0$. The *Hanf number* of $L_{\omega_1\omega_1}$ is

$$h_{\omega_1\omega_1} = \sup\{\sup(\phi) \mid \phi \in L_{\omega_1\omega_1}\}.$$

It is easily seen that $h_{\omega_1\omega_1}$ is a singular strong limit cardinal. The following elementary and well-known lemma is equally obvious (one considers the sentence "I am a wellfounded model of ZFC without inaccessible (or respective) cardinals").

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LEMMA 1. *$h_{\omega_1\omega_1}$ exceeds the first inaccessible (hyperinaccessible, Mahlo, hyper-Mahlo) cardinal (if any).*

J. Silver proved that the predicate $\kappa \to (\omega)^{<\omega}$ is $\Pi_1^{ZFC}$ thereby extending the proof of Lemma 1 to a proof of (see [9])

LEMMA 2. *If $\kappa(\omega)$ exists, then $h_{\omega_1\omega_1} > \kappa(\omega)$. Moreover, if $\kappa(\alpha)$ exists for all $\alpha < \omega_1$, then $h_{\omega_1\omega_1} > \sup\{\kappa(\omega)|\alpha < \omega_1\}$.*

Similarly, $h_{\omega_1\omega_1}$ exceeds the first subtle cardinal (if any).

The set-theoretical notation of the next two sections follows that of [3].

## 3. The main result.

THEOREM 1. *If the theory*

$$\text{ZFC} + \text{“}\textit{there is a proper class of weakly compact cardinals}\text{”}$$

*is consistent, then so is the theory*

$$\text{ZFC} + \text{“}h_{\omega_1\omega_1} < \textit{the first weakly compact cardinal}\text{”}.$$

PROOF. Let us assume that $V = L$ and that $\rho_\alpha$ ($\alpha \in On$) is the proper class of all weakly compact cardinals in ascending order. We may assume that $\sup\{\rho_\alpha|\alpha < \nu\}$ is singular for every limit $\nu$ (otherwise we move into a submodel of the universe).

Let $\mathbf{P}$ be the notion of forcing which results if we add, via backward Easton, a Cohen-subset for every inaccessible cardinal. Thus $\mathbf{P}$ is the direct limit of a system $\{\mathbf{P}_\alpha|\alpha \in On\}$ of notions of forcing defined as follows: For inaccessible $\nu$, $\mathbf{P}_\nu$ is the direct limit of $\{\mathbf{P}_\alpha|\alpha < \nu\}$, and for other limit $\nu$ the corresponding inverse limit. At successor stages we let $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathbf{C}$, where for inaccessible $\alpha$, $\mathbf{C}$ is the notion of forcing for adding a Cohen subset for $\alpha$, and for other $\alpha$, $\mathbf{C} = \{\varnothing\}$. For details concerning backward Easton forcing the reader is referred to [3], [4] and [8].

Following the idea of the proof of Theorem 88 in [3] (or §25 in [4]) one can prove that $\mathbf{P}$ preserves weak compactness. On the other hand, no new cardinals become weakly compact, because weak compactness relativizes to $L$.

K. Kunen describes in [6] a notion of forcing $\mathbf{R}$ which can be used to add a $\kappa$-Souslin tree $T$ for a regular uncountable cardinal $\kappa$. This notion of forcing has the additional property that if $T$ is later killed by forcing with its reverse ordering $\mathbf{T}$ the resulting product forcing $\mathbf{R} * \mathbf{T}$ is isomorphic to the notion of forcing used for adding a Cohen subset for $\kappa$. This fact has a basic role in our proof.

We shall now extend $\mathbf{P}$ by a new backward Easton construction. For this end, let $\mathbf{Q}_0 = \mathbf{P}$. For limit $\nu$, let $\mathbf{Q}_\nu$ be the inverse limit of $\{\mathbf{Q}_\alpha|\alpha < \nu\}$. At successor stages we let $\mathbf{Q}_{\alpha+1} = \mathbf{Q}_\alpha * \mathbf{R}_\alpha$, where $\mathbf{R}_\alpha$ is the notion of forcing, cited above, for adding a $\rho_\alpha$-Souslin tree $T_\alpha$ for $\rho_\alpha$. Finally, let $\mathbf{Q}$ be the direct limit of $\{\mathbf{Q}_\alpha|\alpha \in On\}$. Note that forcing with $\mathbf{Q}_\alpha$ kills the weak compactness of $\rho_\beta$ ($\beta < \alpha$) because of the tree $T_\beta$. On the other hand, $\mathbf{Q}_\alpha$ is mild for $\rho_\alpha$ (as we have no regular limits of weakly compact cardinals) and therefore preserves $\rho_\alpha$ weakly compact, making it in fact the first weakly compact cardinal.

Our third backward Easton construction kills the trees $T_\alpha$ ($\alpha \in On$) one by one. Let $\mathbf{U}_0 = \mathbf{Q}$. For limit $\nu$, let $\mathbf{U}_\nu$ be the inverse limit of $\{\mathbf{U}_\alpha | \alpha < \nu\}$. At successor stages we let $\mathbf{U}_{\alpha+1} = \mathbf{U}_\alpha * T_\alpha$, where $\mathbf{T}_\alpha$ is the reverse ordering of $T_\alpha$. Finally let $\mathbf{U}$ be the direct limit of $\{\mathbf{U}_\alpha | \alpha \in On\}$.

Using the facts that (1) $\mathbf{U}_\alpha$ is isomorphic to the backward Easton extension of $\mathbf{P}$ which adds at first $\mathbf{R}_\beta$ and $\mathbf{T}_\beta$ for $\beta < \alpha$ and then $\mathbf{R}_\beta$ for $\beta \geqslant \alpha$, (2) $\mathbf{R}_\beta * \mathbf{T}_\beta$ is isomorphic to the set of Cohen conditions for $\rho_\beta$, and (3) adding two Cohen subsets is (up to $\cong$) the same as adding just one Cohen subset, one can construct an isomorphism $i$: $\mathbf{U} \cong \mathbf{P}$.

We claim that for some $\alpha$ and for some $p \in \mathbf{Q}_\alpha$ we have

$$p \Vdash_{\mathbf{Q}_\alpha} h_{\omega_1\omega_1} < \check{\rho}_\alpha.$$

This will prove the theorem, as $\rho_\alpha$ is the smallest weakly compact cardinal in any extension by $\mathbf{Q}_\alpha$.

Suppose the claim fails. Then for every $\alpha$ there is a $p_\alpha \in \mathbf{Q}_\alpha$ such that

$$p_\alpha \Vdash_{\mathbf{Q}_\alpha} \exists \phi_\alpha \in L_{\omega_1\omega_1}(\check{\rho}_\alpha \leqslant \sup(\phi_\alpha)).$$

Note that for all notions of forcing used are $< \rho_0$-distributive. Therefore we may pick a $\phi_\alpha \in L_{\omega_1\omega_1}$ such that

$$p_\alpha \Vdash_{\mathbf{Q}_\alpha} \check{\rho}_\alpha \leqslant \sup(\check{\phi}_\alpha). \tag{$*$}$$

By a cardinality argument there is a single sentence $\phi_\alpha \in L_{\omega_1\omega_1}$ such that for arbitrarily large $\beta$

$$p_\beta \Vdash_{\mathbf{Q}_\beta} \check{\rho}_\beta \leqslant \sup(\check{\phi}_\alpha).$$

Using the preservation of the satisfaction relation of $L_{\omega_1\omega_1}$ under extensions not adding new countable sets of ordinals, we can find for arbitrarily large $\beta$ a condition $q_\beta \in \mathbf{U}$ such that

$$q_\beta \Vdash_{\mathbf{U}} \mathrm{Sp}(\check{\phi}_\alpha) - \check{\rho}_\beta \neq \varnothing.$$

By the isomorphism of $\mathbf{U}$ and $\mathbf{P}$, and by the homogeneity of $\mathbf{P}$ (see **[8]**) we obtain

$$1 \Vdash_{\mathbf{P}} \mathrm{Sp}(\check{\phi}_\alpha) - \check{\rho}_\beta \neq \varnothing$$

for arbitrarily large $\beta$ (1 is the maximal element of $\mathbf{P}$). Therefore $1 \Vdash_{\mathbf{P}} \sup(\check{\phi}_\alpha) = 0$, and by another absoluteness argument,

$$1 \Vdash_{\mathbf{Q}_\alpha} \sup(\check{\phi}_\alpha) = 0.$$

This contradicts ($*$), and therefore establishes the claim. This ends the proof of Theorem 1.

**4. Generalizations and related results.** Examination of the proof of Theorem 1 reveals that $L_{\omega_1\omega_1}$ can be considerably extended without affecting the result. The proof works equally well for example for $L_{\omega_2\omega_2}$. More interestingly, we can add new quantifiers to $L_{\omega_1\omega_1}$. A simple quantifier not definable even in $L_{\infty\infty}$ is the Härtig-quantifier

$$IxyA(x)B(y) \quad \text{if and only if } |A(\cdot)| = |B(\cdot)|.$$

Theorem 1 remains true if this quantifier be added to $L_{\omega_1\omega_1}$. This is noteworthy because the Hanf number of the extension of $L_{\omega_1\omega_1}$ by $I$ can be even bigger than a supercompact cardinal (a proof of this will appear elsewhere).

It is apparent that the proof of Theorem 1 extends immediately to a proof of

THEOREM 2. *If the theory*

ZFC + "*there is a proper class of weakly compact cardinals*"

*is consistent, then so is the theory*

> ZFC + *the sentence "the first weakly compact cardinal exceeds the Hanf number of the extension of $L_{\omega_1\omega_1}$ by $Q$" for every generalized quantifier $Q$ the definition of which is provably absolute with respect to extensions which preserve cardinals and cofinalities.*

The role played by weakly compact cardinals in Theorem 1 is by no means unique. For example, only slight changes are needed to prove

THEOREM 3. *If the theory*

ZFC + "*there is a proper class of measurable cardinals*"

*is consistent, then so is the theory*

ZFC + "$h_{\omega_1\omega_1} <$ *the first measurable cardinal*".

T. Jech has pointed out that Ramsey cardinals are also preserved under forcing with **P**. Thus Theorem 1 holds if "weakly compact" is replaced by "Ramsey". This answers problem 9 of H. Friedman [**2**].

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