SHRINKING DECOMPOSITIONS OF $E^n$  
WITH COUNTABLY MANY 1-DIMENSIONAL,  
STAR-LIKE EQUVALENT NONDEGENERATE ELEMENTS  

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ABSTRACT. It is shown that an upper semicontinuous decomposition of $E^n$ ($n > 1$)  
with countably many 1-dimensional, star-like equivalent nondegenerate elements is  
shrinkable.

In this paper we address the problem of shrinking countable decompositions of  
$E^n$ with star-like equivalent nondegenerate elements. Specifically, we prove that if  
$G$ is an upper semicontinuous decomposition of $E^n$ ($n > 1$) such that the collection  
$H_G$ of nondegenerate elements is a countable collection of 1-dimensional, star-like  
equivalent sets, then $E^n/G \cong E^n$. If the nondegenerate elements are tame (i.e.  
1-LCC embedded), then, for $n > 5$, the theorem follows easily from recent results  
of R. D. Edwards [Ed]. However, in general, star-like sets need not be tame.

R. H. Bing [Bi 1] has shown that if $H_G$ is a countable collection of star-like sets,  
then $E^n/G \cong E^n$ ($n > 1$). R. J. Bean [Be] proved that $E^n/G \cong E^n$ ($n > 1$) if $H_G$  
is a null sequence of star-like equivalent sets. The general case where $H_G$ is a  
countable collection of star-like equivalent sets is not known for $n > 3$. (For $n = 1$  
the result is trivial and for $n = 2$ it follows from classical results.) The special case  
when $H_G$ is a countable collection of tame $n$-cells is of interest. (An $n$-cell in $E^n$  
is tame if it is ambiently homeomorphic to a standardly embedded cell.) A recent  
theorem of Starbird and Woodruff [S-W] states that $E^3/G \cong E^3$ if $H_G$ is a  
countable collection of tame 3-cells. The analogous theorem is not known for $n > 4$. R. H. Bing [Bi 1] has shown that $E^n/G \cong E^n$ ($n > 1$) if $H_G$ is a countable  
collection of tame arcs. The theorem presented here is proved using techniques  
similar to those used by Bing. The key is the 1-dimensionality of the (star-like  
equivalent) nondegenerate elements.

Let $X$ be a nonempty compact set in $E^n$ and $p \in X$. The set $X$ is star-like with  
respect to $p$ if each geometric ray emanating from $p$ intersects $X$ in a connected set.  
Equivalently, if $x \in X$, $x \neq p$, then the straight line segment determined by $x$ and $p$  
is contained in $X$. Generally, $X$ is star-like if it is star-like with respect to one of its  
points and star-like equivalent if it is ambiently homeomorphic to a star-like set.

**Main Theorem.** If $G$ is an upper semicontinuous decomposition of $E^n$ ($n > 1$) such  
that $H_G$ is a countable collection of 1-dimensional, star-like equivalent sets, then  
$E^n/G \cong E^n$.

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The theorem follows from Lemma 2 below, where we show that the decomposition \( G \) is shrinkable. In this setting \( G \) is shrinkable if given an open set \( U \) containing the union of the nondegenerate elements and \( \varepsilon > 0 \), then there is a homeomorphism of \( E^n \) onto itself which is the identity on \( E^n - U \) and such that the image of each nondegenerate element has diameter less than \( \varepsilon \). The classical Bing Shrinkability Criterion (see [Bi 1], [Bi 2]) allows us to conclude that the quotient map \( \pi: E^n \to E^n/G \) is approximable by homeomorphisms. Using standard techniques for shrinking decompositions with countably many nondegenerate elements we need only show that a single nondegenerate element can be shrunk without “stretching” other nondegenerate elements. That is, it suffices to verify Lemma 1.

**Lemma 1.** If \( g_0 \in H_G, \varepsilon > 0 \) and \( W \) is a neighborhood of \( g_0 \) in \( E^n \), then there is a homeomorphism \( h: E^n \to E^n \) such that

(a) \( h \) is the identity on \( E^n - W \),
(b) \( \text{diam } h(g_0) < \varepsilon \),
(c) if \( g \in H_G \), then either \( \text{diam } h(g) < \varepsilon \) or \( h(g) \subset N_\varepsilon(g) \), where \( N_\varepsilon(g) \) denotes the \( \varepsilon \)-neighborhood of \( g \) in \( E^n \).

For \( g_0 \in H_G \), let \( \theta: E^n \to E^n \) be a homeomorphism with \( \theta(g_0) \) star-like. Assuming that \( \overline{W} \) is compact and appealing to the uniform continuity of \( \theta^{-1} \) restricted to \( \theta(W) \), Lemma 1 is a consequence of the next result.

**Lemma 2.** If \( G \) is a monotone upper semicontinuous decomposition of \( E^n \), \( g_0 \in H_G \), \( g_0 \) is a 1-dimensional star-like set, \( \varepsilon > 0 \) and \( W \) is a neighborhood of \( g_0 \) in \( E^n \), then there exists a homeomorphism \( h: E^n \to E^n \) such that

(1) \( h \) is the identity on \( E^n - W \),
(2) \( \text{diam } h(g_0) < \varepsilon \),
(3) if \( g \in H_G \), then either \( \text{diam } h(g) < \varepsilon \) or \( h(g) \subset N_\varepsilon(g) \).

**Proof of Lemma 2.** The homeomorphism \( h \) is constructed first to satisfy (1) and (2) using techniques found in [Bi 1]. This construction is outlined below. Condition (3) will follow from a careful replacement of the neighborhood \( W \) which relies heavily on the 1-dimensionality of \( g_0 \).

Let \( g_0 \) be star-like with respect to \( x_0 \). For \( i > 1 \) let \( C_i \) be the round ball of radius \( ie/4 \) centered at \( x_0 \). Let \( m \) be the least positive integer such that \( g_0 \subset \text{int } C_m \). Since \( g_0 \) is star-like it has a neighborhood system, \( \{D_i\} \), of \( n \)-cells which are ideally star-like with respect to \( x_0 \). That is, \( \text{Bd } D_i \) is a tame \((n-1)\)-sphere and each geometric ray emanating from \( x_0 \) pierces \( \text{Bd } D_i \) in exactly one point (see [Bi 1, Lemma 3]). Using upper semicontinuity we choose from this system cells \( D_i, 1 < i < m \), such that

(a) \( g_0 = D_0 \subset \text{int } D_1 \subset D_1 \subset \cdots \subset \text{int } D_m \subset D_m \subset C_m \cap W \),
(b) if \( g \in H_G \) and \( g \cap \text{Bd } D_i \neq \emptyset \) then \( g \cap D_{i-1} = \emptyset, 1 < i < m \).

We define the action of \( h \) on each geometric ray \( R \) emanating from \( x_0 \). Let \( y_0 = x_0 \) and for \( 1 < i < m \) let \( x_i = R \cap \text{Bd } C_i \) and \( y_i = R \cap \text{Bd } D_i \). Let \( f \) be the map of \( R \cap D_m \) onto \( R \cap C_m \) such that \( f(y_i) = x_i, 0 < i < m \), and taking the
segments $y_iy_{i+1}$ linearly onto the segments $x_ix_{i+1}$, $0 < i < m - 1$. Define $h$ to be
the identity on $R - D_m$ and for $x \in R \cap D_m$ let $h(x)$ be the nearer to $x_0$ of $x, f(x)$. 
This is precisely the homeomorphism described in [Bi 1, Lemma 4].

To see that (1) is satisfied observe that $h$ is the identity outside $D_m$ and $D_m \subset W$. 
Clearly $h(D_1) \subset C_1$ and consequently $\text{diam } h(g_0) < \text{diam } C_1 = \varepsilon/2$ and (2) is 
satisfied. One important feature of $h$ should be isolated.

If $x \in D_i - D_{i-1}$ and $h(x) \neq x$, then $h(x) \in C_i - C_{i-1}$. (*)

We now specify the restrictions on $W$ that ensure (3). If $g_0 \subset C_1$ then no 
replacement for $W$ is needed since $h = \text{id}$. Assume that $g_0 \not\subset C_1$. Since $g_0$ is 
1-dimensional and star-like, $g_0 \cap \text{Bd } C_1$ is compact and 0-dimensional. Find a 
pairwise disjoint collection $U_1, \ldots, U_r$ of open subsets of $\text{Bd } C_1$ which cover 
$g_0 \cap \text{Bd } C_1$ and such that $\text{diam } \bar{U}_j < \varepsilon/2$, $1 < j < r$, where $\bar{U}_j$ denotes the radical 
projection of $U_j$ from $x_0$ onto $\text{Bd } C_m$. Let $V_j$ be the union of the straight line 
segments connecting $x_0$ with points of $U_j$; i.e. the geometric cone over $\bar{U}_j$ from $x_0$. 
It follows that $V = (\bigcup_{j=1}^r V_j) \cup C_1$ is a neighborhood of $g_0$. We insist that $W$ be 
contained in $V$. The crucial feature of $W$ is that for $2 < i < m$ each component of 
$W \cap [C_i - C_{i-1}]$ has diameter less than $\varepsilon$.

To see that this restriction on $W$ forces $h$ to satisfy (3) suppose $g \in H_G$. Either 
there is an index $i$ so that $g \subset D_i - D_{i-2}$ or $g \subset E^n - D_{m-1}$. In either case, if 
$K = \{x \in g|h(x) \neq x\}$, then (*) shows that for some index $i$, $h(K) \subset C_i - C_{i-2}$. If 
$K = \emptyset$, then $h(g) = g \subset N_i(g)$. If $K \neq \emptyset$ and $A$ is a component of $K$, then $h(A)$ is 
contained in some component of $W \cap [C_i - C_{i-2}]$ and $\text{diam } h(A) < \varepsilon$. If $A = g$, 
then $\text{diam } g < \varepsilon$. Otherwise, since $g$ is connected, there exists a point $x \in g \cap 
F(h(A))$ from which $h(A) \subset N_i(g)$. It follows that $h(g) \subset N_i(g)$. This establishes 
(3) and completes the proof of Lemma 2 and the theorem.

References

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