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PROPERTIES OF $\beta X - X$ FOR LOCALLY CONNECTED GENERALIZED CONTINUA

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ABSTRACT. Let X be a locally connected generalized continuum and let βX denote the Stone-Čech compactification of X. In this paper are given necessary and sufficient conditions for $\beta X - X$ to be the union of a finite number of disjoint continua, and for each of these continua to be indecomposable.

1. Introduction. R. F. Dickman has characterized those locally connected generalized continua X for which $\beta X - X$ is an indecomposable continuum in terms of the so-called strong complementation property [3]. He also observed that $\beta X - X$ is a continuum if and only if X has the complementation property. A simple consequence of the work of K. D. Magill is that the locally connected generalized continua with the complementation property are precisely those having the one-point compactification as the only finite compactification [6]. This observation suggests that perhaps certain properties characterizing spaces with n-point compactifications might also provide extensions of Dickman's results. Here we do, in fact, characterize those locally connected generalized continua X for which $\beta X - X$ has exactly n components, k of which are indecomposable. These characterizations are given in terms of an n-complementation property and a strong n-complementation property which contain the complementation property and the strong complementation property as special cases.

2. Preliminaries. We shall assume throughout that the space X is separable and metric. A *continuum* is a compact connected space and a *generalized continuum* is a locally compact connected space. A continuum is called *indecomposable* if it is not the union of two proper subcontinua, and a set is called *conditionally compact* if its closure is compact.

As usual, βX denotes the Stone-Čech compactification of X. A compactification αX is said to be finite if $\alpha X - X$ consists of a finite number of points, and in particular is called an n-point compactification if $\alpha X - X$ consists of exactly n points.

LEMMA 2.1. *Suppose X is locally connected and locally compact, and αX is a finite compactification of X. If $r \in \alpha X - X$ and $U \subset X$ is an open set such that $r \notin \mathrm{cl}_{\alpha X}(X - U)$, then for any open $V \subset U$ for which $U - V$ is conditionally compact in X, $V \cup \{r\}$ is an αX-open neighborhood of r.*

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PROOF. If $V \cup \{r\}$ is not an αX neighborhood of r, then r is an accumulation point of $\alpha X - V$, and hence of $U - V$. This contradicts the conditional compactness in X of $U - V$.

PROPOSITION 2.2. *If X is a locally connected generalized continuum and αX is a finite compactification of X, then αX is locally connected and metric.*

PROOF. That αX is metric is an immediate consequence of Lemma 2 of Magill [5]. It follows from (12.3), page 19 of Whyburn [7] that the continuum αX cannot fail to be locally connected on only a finite set.

3. The n-complementation property.

THEOREM 3.1. *If X is locally compact, then $\beta X - X$ has at least n components if and only if X has an n-point compactification.*

PROOF. First, suppose $\beta X - X$ has at least n components. Now X is an open subset of the compact space βX and its complement has at least n components, so it follows from Theorem 2.1 of Cain [2] that X has an n-point compactification.

Next, suppose X has an n-point compactification αX. It follows at once that $\beta X - X$ has at least n components since $\alpha X - X$ is a continuous image of $\beta X - X$ (Gillman and Jerison [4, Theorem 6.2, p. 92]).

COROLLARY 3.2. *The remainder $\beta X - X$ is a continuum if and only if the one-point compactification is the only finite compactification of X.*

DEFINITION 3.3. A space X is said to have the n-complementation property if for each compact set $A \subset X$ there is a compact set K so that $A \subset K$ and $X - K$ has exactly n nonconditionally compact components.

This concept generalizes Dickman's complementation property in that a noncompact space X has the 1-complementation property if and only if it has the complementation property.

THEOREM 3.4. *If X is a locally connected generalized continuum, then it has the n-complementation property if and only if it has a k-point compactification for every $k \leq n$, and does not have a k-point compactification of $k > n$.*

PROOF. Suppose X has the n-complementation property. Then there is a compact $K \subset X$ so that $X - K = C_1 \cup C_2 \cup \ldots \cup C_n$, where the C_i are open, nonconditionally compact, and mutually disjoint. The existence of a k-point compactification for $k \leq n$ is an almost immediate consequence of Theorem (2.1) of Magill [6]. We need only note that $K \cup C_i$ is noncompact for each i, since $\operatorname{cl} C_i$ is noncompact. The nonexistence of a k-point compactification for $k > n$ is guaranteed by Theorem (2.6) of Magill [6].

Now assume X has an n-point compactification αX but does not have an $(n + 1)$-point compactification, and let $A \subset X$. If $\alpha X - X = \{r_1, r_2, \ldots, r_n\}$, choose mutually disjoint open neighborhoods of the r_i, say $U_1, U_2, \ldots, U_n$, in such a way that $U_i \cap A = \varnothing$ for each i. Then $K = X - \bigcup_1^n U_i$ is compact, contains A, and $X - K$ has at least n nonconditionally compact components. The space X is

locally connected, so $X - K$ has finitely many components. If it were to have more than n nonconditionally compact components, it would follow, as argued previously, that X would have a finite compactification of more than n points. Thus X has the n-complementation property.

The next theorem is an immediate consequence of the previous two.

THEOREM 3.5. *If X is a locally connected generalized continuum, then $\beta X - X$ is the union of exactly n disjoint continua if and only if X has the n-complementation property.*

COROLLARY 3.6 (DICKMAN). *If X is a locally connected generalized continuum, then $\beta X - X$ is a continuum if and only if X has the complementation property.*

COROLLARY 3.7. *For $m > 1$, $\beta \mathbf{R}^m - \mathbf{R}^m$ is a continuum, and $\beta \mathbf{R}^1 - \mathbf{R}^1$ is the union of two disjoint continua.*

4. The strong n-complementation property.

DEFINITION 4.1. A space X is said to have the strong n-complementation property if it contains a compact set K whose complement has at least n nonconditionally compact components and for every collection $\{G_1, G_2, \ldots, G_n\}$ of n mutually disjoint open connected, nonconditionally compact sets, it is true that $X - \bigcup_1^n G_i$ is compact.

This concept generalizes the strong complementation property of Dickman [3].

PROPOSITION 4.2. *If X is locally connected and has the strong n-complemenation property, then it has the n-complementation property.*

PROOF. Let $A \subset X$ be compact. There is a compact $C \subset X$, so that $X - C$ has at least n nonconditionally compact components. Let $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_n$ be n of them. For each i, define $\tilde{G}_i = \hat{G}_i - A$. Each of the $\tilde{G}_i$ is nonempty, open and nonconditionally compact and thus has a nonconditionally compact component G_i. Hence $X - \bigcup_1^n G_i$ is compact, and $A \subset X - \bigcup_1^n G_i$.

THEOREM 4.3. *If X is a locally connected generalized continuum, then $\beta X - X$ is the disjoint union of exactly n indecomposable continua if and only if X has the strong n-complementation property.*

PROOF. First, suppose X has the strong n-complementation property. Then X has a compactification αX so that $\alpha X - X = \{r_1, r_2, \ldots, r_n\}$, and from Proposition 2.2 we know that αX is locally connected and metric. Choose connected sets $U_1, U_2, \ldots, U_n$ which are open in αX, have mutually disjoint closures in αX, and have $r_i \in U_i$ for each i. Each $\mathrm{cl}_{\alpha X}(U_i)$ is arcwise connected, so for $z_i \in U_i \cap X$, there is an arc I_i in U_i between z_i and r_i. Thus each $R_i = I_i - \{r_i\}$ is a ray in X and $R_i \cap R_j = \varnothing$ for $i \neq j$.

The R_i's are closed in the normal space X, so $\mathrm{cl}_{\beta X} R_i$ is homeomorphic to βR_i (6.9, p. 89 of Gillman and Jerison [4]). We know that for a ray R, $\beta R - R$ is an indecomposable continuum (Bellamy [1]). Letting $R_i^* = \beta R_i - R_i = \mathrm{cl}_{\beta X} R_i - R_i$, we have $\bigcup_1^n R_i^* \subset \beta X - X$, and each R_i^* is an indecomposable continuum. It remains to show that $\bigcup_1^n R_i^* = \beta X - X$.

Suppose there is a $p \in \beta X - X$ and $p \notin \bigcup_1^n R_i^*$. Then there are disjoint βX-open sets U and V such that $p \in U$ and $\mathrm{cl}_{\beta X} R_i \subset V$. Now $V \cap X$ is an X-open neighborhood of $\bigcup_1^n R_i$, so there are mutually disjoint open connected sets in X, say $G_1, G_2, \ldots, G_n$, which are such that $R_i \subset G_i \subset V \cap X$. We know from the strong n-complementation property that $X - \bigcup_1^n G_i$ is compact, clearly contradicting the fact that $p \in \mathrm{cl}_{\beta X}(X - \bigcup_1^n G_i)$. Thus $\bigcup_1^n R_i^* = \beta X - X$.

Now, suppose $\beta X - X = \bigcup_1^n C_i$, where each C_i is an indecomposable continuum and $C_i \cap C_j = \varnothing$ for $i \neq j$. It follows from Theorem 3.1 that X has an n-point compactification αX. If X does not have the strong n-complementation property, then there exist mutually disjoint open connected, nonconditionally compact sets $G_1, G_2, \ldots, G_n$ such that $X - \bigcup_1^n G_i$ is not compact. This means that at least one $r \in \alpha X - X$ is an accumulation point of $X - \bigcup_1^n G_i$. Every $p \in \alpha X - X$ is an accumulation point of $\bigcup_1^n G_i$; otherwise there would be an αX neighborhood of p, say N, such that $(N \cap X) \cap G_i = \varnothing$ for every i, and this would imply the existence of an $(n+1)$-point compactification X, which cannot be, according to Theorem 3.1. It follows that r is an accumulation point of some G_k, and also of $X - G_k$.

Let $\{U_m\}$ be a sequence of αX-open, connected neighborhoods of r such that $U_{m+1} \subset U_m$ and $\{r\} = \bigcap_1^\infty U_m$. For each m, let K_m be the component of $U_m \cap \mathrm{cl}_{\alpha X}(G_k)$ containing r. Then $K_{m+1} \subset K_m$. Now let $\mathcal{C}$ be the collection of all components of $U_m - K_m$ and define $\mathcal{V} = \{V \in \mathcal{C} : r \in \mathrm{cl}_{\alpha X}(V)\}$. Thus

$$U_m - K_m = [\bigcup \mathcal{V}] \cup [\bigcup (\mathcal{C} - \mathcal{V})].$$

The set $K_m \cup [\bigcup(\mathcal{C} - \mathcal{V})]$ is connected.

Define $A_m = \mathrm{cl}_{\alpha X}[[\bigcup(\mathcal{C} - \mathcal{V})] \cup K_m]$, and $B_m = \mathrm{cl}_{\alpha X}(\bigcup \mathcal{V})$. Then A_m and B_m are nonempty continua, $\mathrm{cl}_{\alpha X} U_m = A_m \cup B_m$, and $\{A_m\}$, $\{B_m\}$ are decreasing sequences.

There is a continuous function f: $\beta X \to \alpha X$ so that $f|X$ is the identity and $f(\beta X - X) = \alpha X - X$ (Theorem 6.12, p. 92, Gillman and Jerison [4]). Thus f is monotone, so $f^{-1}(A_m)$ and $f^{-1}(B_m)$ are connected, and $\{f^{-1}(A_m)\}$ and $\{f^{-1}(B_m)\}$ are decreasing sequences of continua in βX. It follows that $\bigcap_1^\infty f^{-1}(A_m)$ and $\bigcap_1^\infty f^{-1}(B_m)$ are nonempty continua, and since $\mathrm{cl}_{\alpha X} U_m = A_m \cup B_m$, it follows that $[\bigcap_1^\infty f^{-1}(A_m)] \cup [\bigcap_1^\infty f^{-1}(B_m)] = f^{-1}(r) = C_r$, a component of $\beta X - X$. Each component of $\beta X - X$ is, however, indecomposable so this means that $C_r = \bigcap_1^\infty f^{-1}(A_m) = \bigcap_1^\infty f^{-1}(B_m)$. This implies that A_m is a neighborhood of r and that B_m is a neighborhood of r, which is a contradiction. Hence it must be true that X has the strong n-complementation property.

COROLLARY 4.4. *If $A = [0, +\infty)$ and X is a locally connected generalized continuum, then $\beta X - X$ is the union of n disjoint continua each homeomorphic to $\beta A - A$ if and only if X has the strong n-complementation property.*

COROLLARY 4.5 (DICKMAN). *If X is a locally connected generalized continuum, then $\beta X - X$ is an indecomposable continuum if and only if X has the strong complementation property.*

The following theorem covers the situation in which possibly some components of $\beta X - X$ are decomposable and some are indecomposable.

THEOREM 4.6. *Suppose X is a locally connected generalized continuum and $\beta X - X$ consists of the union of n disjoint continua. At least k of these components of $\beta X - X$ are indecomposable if and only if there is an open set $U \subset X$ such that the boundary of U is compact and $X - U$ has the strong k-complementation property.*

PROOF. Let αX be a compactification of X with $\alpha X - X = \{r_1, r_2, \ldots, r_n\}$.

First, assume $U \subset X$ is open, $\mathrm{Fr}(U)$ is compact, and $X - U$ has the strong k-complementation property. It follows from Lemma 2.1 that for $n - k$ of the r_i, say $r_{k+1}, \ldots, r_n$, it is true that $U \cup \{r_{k+1}, \ldots, r_n\}$ is open in αX. Each r_i is a noncut point of αX, so the existence of an αX open neighborhood W of $\{r_{k+1}, \ldots, r_n\}$ such that $W \subset U \cup \{r_{k+1}, \ldots, r_n\}$ and $\alpha X - W$ is connected follows from (4.15) on page 50 of Whyburn [7]. Thus $A = X - W$ is a locally connected generalized continuum, and it has the strong k-complementation property. This means that $\beta A - A$ consists of the disjoint union of exactly k indecomposable continua. But βA is homeomorphic with $\mathrm{cl}_{\beta X} A$, so the existence of k indecomposable continua among the components of $\beta X - X$ follows.

Next, suppose at least k of the components of $\beta X - X$ are indecomposable. Let $f\colon \beta X \to \alpha X$ be a map such that $f|X$ is the identity and $f(\beta X - X) = \alpha X - X$. Let $\{r_1, \ldots, r_k\}$ denote the image of the k indecomposable components. As argued in the previous paragraph, there is an αX neighborhood W of $\{r_{k+1}, \ldots, r_n\}$ so that $A = X - W$ is a locally connected generalized continuum, and $\beta A - A$ is the disjoint union of k indecomposable continua. Thus $A = X - W$ has the strong k-complementation property. It is clear that the boundary of $U = W \cap X$ is compact, so the proof is complete.

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