PROPERTIES OF $\beta X - X$ FOR LOCALLY CONNECTED GENERALIZED CONTINUA

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Abstract. Let $X$ be a locally connected generalized continuum and let $\beta X$ denote the Stone-Cech compactification of $X$. In this paper are given necessary and sufficient conditions for $\beta X - X$ to be the union of a finite number of disjoint continua, and for each of these continua to be indecomposable.

1. Introduction. R. F. Dickman has characterized those locally connected generalized continua $X$ for which $\beta X - X$ is an indecomposable continuum in terms of the so-called strong complementation property [3]. He also observed that $\beta X - X$ is a continuum if and only if $X$ has the complementation property. A simple consequence of the work of K. D. Magill is that the locally connected generalized continua with the complementation property are precisely those having the one-point compactification as the only finite compactification [6]. This observation suggests that perhaps certain properties characterizing spaces with $n$-point compactifications might also provide extensions of Dickman's results. Here we do, in fact, characterize those locally connected generalized continua $X$ for which $\beta X - X$ has exactly $n$ components, $k$ of which are indecomposable. These characterizations are given in terms of an $n$-complementation property and a strong $n$-complementation property which contain the complementation property and the strong complementation property as special cases.

2. Preliminaries. We shall assume throughout that the space $X$ is separable and metric. A continuum is a compact connected space and a generalized continuum is a locally compact connected space. A continuum is called indecomposable if it is not the union of two proper subcontinua, and a set is called conditionally compact if its closure is compact.

As usual, $\beta X$ denotes the Stone-Čech compactification of $X$. A compactification $\alpha X$ is said to be finite if $\alpha X - X$ consists of a finite number of points, and in particular is called an $n$-point compactification if $\alpha X - X$ consists of exactly $n$ points.

Lemma 2.1. Suppose $X$ is locally connected and locally compact, and $\alpha X$ is a finite compactification of $X$. If $r \in \alpha X - X$ and $U \subset X$ is an open set such that $r \notin \text{cl}_{\alpha X}(X - U)$, then for any open $V \subset U$ for which $U - V$ is conditionally compact in $X$, $V \cup \{r\}$ is an $\alpha X$-open neighborhood of $r$.
PROOF. If \( V \cup \{ r \} \) is not an \( \alpha X \) neighborhood of \( r \), then \( r \) is an accumulation point of \( \alpha X - V \), and hence of \( U - V \). This contradicts the conditional compactness in \( X \) of \( U - V \).

PROPOSITION 2.2. If \( X \) is a locally connected generalized continuum and \( \alpha X \) is a finite compactification of \( X \), then \( \alpha X \) is locally connected and metric.

PROOF. That \( \alpha X \) is metric is an immediate consequence of Lemma 2 of Magill [5]. It follows from (12.3), page 19 of Whyburn [7] that the continuum \( \alpha X \) cannot fail to be locally connected on only a finite set.

3. The \( n \)-complementation property.

THEOREM 3.1. If \( X \) is locally compact, then \( \beta X - X \) has at least \( n \) components if and only if \( X \) has an \( n \)-point compactification.

PROOF. First, suppose \( \beta X - X \) has at least \( n \) components. Now \( X \) is an open subset of the compact space \( \beta X \) and its complement has at least \( n \) components, so it follows from Theorem 2.1 of Cain [2] that \( X \) has an \( n \)-point compactification.

Next, suppose \( X \) has an \( n \)-point compactification \( \alpha X \). It follows at once that \( \beta X - X \) has at least \( n \) components since \( \alpha X - X \) is a continuous image of \( \beta X - X \) (Gillman and Jerison [4, Theorem 6.2, p. 92]).

COROLLARY 3.2. The remainder \( \beta X - X \) is a continuum if and only if the one-point compactification is the only finite compactification of \( X \).

DEFINITION 3.3. A space \( X \) is said to have the \( n \)-complementation property if for each compact set \( A \subset X \) there is a compact set \( K \) so that \( A \subset K \) and \( X - K \) has exactly \( n \) nonconditionally compact components.

This concept generalizes Dickman's complementation property in that a noncompact space \( X \) has the 1-complementation property if and only if it has the complementation property.

THEOREM 3.4. If \( X \) is a locally connected generalized continuum, then it has the \( n \)-complementation property if and only if it has a \( k \)-point compactification for every \( k < n \), and does not have a \( k \)-point compactification of \( k > n \).

PROOF. Suppose \( X \) has the \( n \)-complementation property. Then there is a compact \( K \subset X \) so that \( X - K = C_1 \cup C_2 \cup \ldots \cup C_n \), where the \( C_i \) are open, nonconditionally compact, and mutually disjoint. The existence of a \( k \)-point compactification for \( k < n \) is an almost immediate consequence of Theorem (2.1) of Magill [6]. We need only note that \( K \cup C_i \) is noncompact for each \( i \), since \( cl C_i \) is noncompact. The nonexistence of a \( k \)-point compactification for \( k > n \) is guaranteed by Theorem (2.6) of Magill [6].

Now assume \( X \) has an \( n \)-point compactification \( \alpha X \) but does not have an \((n + 1)\)-point compactification, and let \( A \subset X \). If \( \alpha X - X = \{ r_1, r_2, \ldots, r_n \} \), choose mutually disjoint open neighborhoods of the \( r_i \), say \( U_1, U_2, \ldots, U_n \), in such a way that \( U_i \cap A = \emptyset \) for each \( i \). Then \( K = X - \bigcup U_i \) is compact, contains \( A \), and \( X - K \) has at least \( n \) nonconditionally compact components. The space \( X \) is
locally connected, so \( X - K \) has finitely many components. If it were to have more than \( n \) nonconditionally compact components, it would follow, as argued previously, that \( X \) would have a finite compactification of more than \( n \) points. Thus \( X \) has the \( n \)-complementation property.

The next theorem is an immediate consequence of the previous two.

**Theorem 3.5.** If \( X \) is a locally connected generalized continuum, then \( \beta X - X \) is the union of exactly \( n \) disjoint continua if and only if \( X \) has the \( n \)-complementation property.

**Corollary 3.6 (Dickman).** If \( X \) is a locally connected generalized continuum, then \( \beta X - X \) is a continuum if and only if \( X \) has the complementation property.

**Corollary 3.7.** For \( m > 1 \), \( \beta \mathbb{R}^m - \mathbb{R}^m \) is a continuum, and \( \beta \mathbb{R}^1 - \mathbb{R}^1 \) is the union of two disjoint continua.

4. **The strong \( n \)-complementation property.**

**Definition 4.1.** A space \( X \) is said to have the strong \( n \)-complementation property if it contains a compact set \( K \) whose complement has at least \( n \) nonconditionally compact components and for every collection \( \{G_1, G_2, \ldots, G_n\} \) of \( n \) mutually disjoint open connected, nonconditionally compact sets, it is true that \( X - \bigcup G_i \) is compact.

This concept generalizes the strong complementation property of Dickman [3].

**Proposition 4.2.** If \( X \) is locally connected and has the strong \( n \)-complementation property, then it has the \( n \)-complementation property.

**Proof.** Let \( A \subset X \) be compact. There is a compact \( C \subset X \), so that \( X - C \) has at least \( n \) nonconditionally compact components. Let \( \hat{G}_1, \hat{G}_2, \ldots, \hat{G}_n \) be \( n \) of them. For each \( i \), define \( \tilde{G}_i = \hat{G}_i - A \). Each of the \( \tilde{G}_i \) is nonempty, open and nonconditionally compact and thus has a nonconditionally compact component \( G_i \). Hence \( X - \bigcup \tilde{G}_i \) is compact, and \( A \subset X - \bigcup \tilde{G}_i \).

**Theorem 4.3.** If \( X \) is a locally connected generalized continuum, then \( \beta X - X \) is the disjoint union of exactly \( n \) indecomposable continua if and only if \( X \) has the strong \( n \)-complementation property.

**Proof.** First, suppose \( X \) has the strong \( n \)-complementation property. Then \( X \) has a compactification \( \alpha X \) so that \( \alpha X - X = \{r_1, r_2, \ldots, r_n\} \), and from Proposition 2.2 we know that \( \alpha X \) is locally connected and metric. Choose connected sets \( U_1, U_2, \ldots, U_n \) which are open in \( \alpha X \), have mutually disjoint closures in \( \alpha X \), and have \( r_i \in U_i \) for each \( i \). Each \( \text{cl}_{\alpha X}(U_i) \) is arcwise connected, so for \( z_i \in U_i \cap X \), there is an arc \( I_i \) in \( U_i \) between \( z_i \) and \( r_i \). Thus each \( R_i = I_i - \{r_i\} \) is a ray in \( X \) and \( R_i \cap R_j = \emptyset \) for \( i \neq j \).

The \( R_i \)'s are closed in the normal space \( X \), so \( \text{cl}_{\beta X} R_i \) is homeomorphic to \( \beta R_i \) (6.9, p. 89 of Gillman and Jerison [4]). We know that for a ray \( R \), \( \beta R - R \) is an indecomposable continuum (Bellamy [1]). Letting \( R_i^* = \beta R_i - R_i = \text{cl}_{\beta X} R_i - R_i \), we have \( \bigcup R_i^* \subset \beta X - X \), and each \( R_i^* \) is an indecomposable continuum. It remains to show that \( \bigcup R_i^* = \beta X - X \).
Suppose there is a \( p \in \beta X - X \) and \( p \notin \bigcup \gamma R^* \). Then there are disjoint \( \beta X \)-open sets \( U \) and \( V \) such that \( p \in U \) and \( \text{cl}_{\beta X} R_i \subset V \). Now \( V \cap X \) is an \( X \)-open neighborhood of \( \bigcup \gamma R_i \), so there are mutually disjoint open connected sets in \( X \), say \( G_1, G_2, \ldots, G_n \), which are such that \( R_i \subset G_i \subset V \cap X \). We know from the strong \( n \)-complementation property that \( X - \bigcup \gamma G_i \) is compact, clearly contradicting the fact that \( p \in \text{cl}_{\beta X} (X - \bigcup \gamma G). \) Thus \( \bigcup \gamma R^*_p = \beta X - X \).

Now, suppose \( \beta X - X = \bigcup \gamma C_i \), where each \( C_i \) is an indecomposable continuum and \( C_i \cap C_j = \emptyset \) for \( i \neq j \). It follows from Theorem 3.1 that \( X \) has an \( n \)-point compactification \( \alpha X \). If \( X \) does not have the strong \( n \)-complementation property, then there exist mutually disjoint open connected, nonconditionally compact sets \( G_1, G_2, \ldots, G_n \) such that \( X - \bigcup \gamma G_i \) is not compact. This means that at least one \( r \in \alpha X - X \) is an accumulation point of \( X - \bigcup \gamma G_i \). Every \( p \in \alpha X - X \) is an accumulation point of \( \bigcup \gamma G_i \); otherwise there would be an \( \alpha X \) neighborhood of \( p \), say \( N \), such that \( (N \cap X) \cap G_i = \emptyset \) for every \( i \), and this would imply the existence of an \( (n + 1) \)-point compactification \( X \), which cannot be, according to Theorem 3.1. It follows that \( r \) is an accumulation point of some \( G_k \), and also of \( X - G_k \).

Let \( \{ U_m \} \) be a sequence of \( \alpha X \)-open, connected neighborhoods of \( r \) such that \( U_{m+1} \subset U_m \) and \( \{ r \} = \bigcap \gamma U_m \). For each \( m \), let \( K_m \) be the component of \( U_m \cap \text{cl}_{\alpha X}(G_k) \) containing \( r \). Then \( K_{m+1} \subset K_m \). Now let \( C \) be the collection of all components of \( U_m - K_m \) and define \( \mathcal{V} = \{ V \in C : r \in \text{cl}_{\alpha X}(V) \} \). Thus

\[
U_m - K_m = [\bigcup \mathcal{V}] \cup [\bigcup (C - \mathcal{V})].
\]

The set \( K_m \cup [\bigcup (C - \mathcal{V})] \) is connected.

Define \( A_m = \text{cl}_{\alpha X} [\bigcup (C - \mathcal{V})] \cup K_m \), and \( B_m = \text{cl}_{\alpha X} \bigcup \mathcal{V} \). Then \( A_m \) and \( B_m \) are nonempty continua, \( \text{cl}_{\alpha X} U_m = A_m \cup B_m \), and \( \{ A_m \}, \{ B_m \} \) are decreasing sequences.

There is a continuous function \( f : \beta X \to \alpha X \) so that \( f|X \) is the identity and \( f(\beta X - X) = \alpha X - X \) (Theorem 6.12, p. 92, Gillman and Jerison [4]). Thus \( f \) is monotone, so \( f^{-1}(A_m) \) and \( f^{-1}(B_m) \) are connected, and \( \{ f^{-1}(A_m) \} \) and \( \{ f^{-1}(B_m) \} \) are decreasing sequences of continua in \( \beta X \). It follows that \( \bigcap \gamma f^{-1}(A_m) \) and \( \bigcap \gamma f^{-1}(B_m) \) are nonempty continua, and since \( \text{cl}_{\alpha X} U_m = A_m \cup B_m \), it follows that \( \bigcap \gamma f^{-1}(A_m) \cup \bigcap \gamma f^{-1}(B_m) = f^{-1}(r) = C_r \), a component of \( \beta X - X \). Each component of \( \beta X - X \) is, however, indecomposable so this means that \( C_r = \bigcap \gamma f^{-1}(A_m) = \bigcap \gamma f^{-1}(B_m) \). This implies that \( A_m \) is a neighborhood of \( r \) and that \( B_m \) is a neighborhood of \( r \), which is a contradiction. Hence it must be true that \( X \) has the strong \( n \)-complementation property.

**Corollary 4.4.** If \( A = [0, +\infty) \) and \( X \) is a locally connected generalized continuum, then \( \beta X - X \) is the union of \( n \) disjoint continua each homeomorphic to \( \beta A - A \) if and only if \( X \) has the strong \( n \)-complementation property.

**Corollary 4.5 (Dickman).** If \( X \) is a locally connected generalized continuum, then \( \beta X - X \) is an indecomposable continuum if and only if \( X \) has the strong complementation property.
The following theorem covers the situation in which possibly some components of $\beta X - X$ are decomposable and some are indecomposable.

**Theorem 4.6.** Suppose $X$ is a locally connected generalized continuum and $\beta X - X$ consists of the union of $n$ disjoint continua. At least $k$ of these components of $\beta X - X$ are indecomposable if and only if there is an open set $U \subset X$ such that the boundary of $U$ is compact and $X - U$ has the strong $k$-complementation property.

**Proof.** Let $\alpha X$ be a compactification of $X$ with $\alpha X - X = \{r_1, r_2, \ldots, r_n\}$.

First, assume $U \subset X$ is open, $\text{Fr}(U)$ is compact, and $X - U$ has the strong $k$-complementation property. It follows from Lemma 2.1 that for $n - k$ of the $r_i$, say $r_{k+1}, \ldots, r_n$, it is true that $U \cup \{r_{k+1}, \ldots, r_n\}$ is open in $\alpha X$. Each $r_i$ is a noncut point of $\alpha X$, so the existence of an $\alpha X$ open neighborhood $W$ of $\{r_{k+1}, \ldots, r_n\}$ such that $W \subset U \cup \{r_{k+1}, \ldots, r_n\}$ and $\alpha X - W$ is connected follows from (4.15) on page 50 of Whyburn [7]. Thus $A = X - W$ is a locally connected generalized continuum, and it has the strong $k$-complementation property. This means that $\beta A - A$ consists of the disjoint union of exactly $k$ indecomposable continua. But $\beta A$ is homeomorphic with $\text{cl}_{\beta X}A$, so the existence of $k$ indecomposable continua among the components of $\beta X - X$ follows.

Next, suppose at least $k$ of the components of $\beta X - X$ are indecomposable. Let $f: \beta X \to \alpha X$ be a map such that $f|X$ is the identity and $f(\beta X - X) = \alpha X - X$. Let $\{r_1, \ldots, r_k\}$ denote the image of the $k$ indecomposable components. As argued in the previous paragraph, there is an $\alpha X$ neighborhood $W$ of $\{r_{k+1}, \ldots, r_n\}$ so that $A = X - W$ is a locally connected generalized continuum, and $\beta A - A$ is the disjoint union of $k$ indecomposable continua. Thus $A = X - W$ has the strong $k$-complementation property. It is clear that the boundary of $U = W \cap X$ is compact, so the proof is complete.

**References**


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