A NOTE ON EXTREMALLY DISCONNECTED SPACES

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Abstract. A topological space $X$ is said to be locally $S$-closed if each point of $X$ has an open neighborhood which is an $S$-closed subspace of $X$. In this note it is shown that every locally $S$-closed weakly Hausdorff (or almost-regular) space is extremally disconnected.

1. Introduction. In 1976, T. Thompson [11] introduced the concept of $S$-closed spaces. Recently, the following sufficient conditions for an $S$-closed space to be extremally disconnected have been obtained.

Theorem A (Herrmann [3]). An $S$-closed weakly Hausdorff space is extremally disconnected.

Theorem B (Herrmann [3]). An $S$-closed almost-regular space is extremally disconnected.

Theorem C (Cameron [1]). A maximal $S$-closed space is extremally disconnected.

In [7], the present author introduced the concept of locally $S$-closed spaces which is strictly weaker than that of $S$-closed spaces. The purpose of the present note is to show that the condition "$S$-closed" in the theorems above stated can be replaced by "locally $S$-closed".

2. Preliminaries. Let $(X, \tau)$ be a topological space and $S$ a subset of $X$. The closure of $S$ and the interior of $S$ in $(X, \tau)$ are denoted by $\text{Cl}_X(S)$ and $\text{Int}_X(S)$, respectively. A subset $S$ of $X$ is said to be regular open (regular closed) if $\text{Int}_X(\text{Cl}_X(S)) = S$ (resp. $\text{Cl}_X(\text{Int}_X(S)) = S$). A topological space $(X, \tau)$ is said to be extremally disconnected if $\text{Cl}_X(U) \in \tau$ for every $U \in \tau$. A subset $S$ of $(X, \tau)$ is said to be semiopen [4] if there exists $U \in \tau$ such that $U \subset S \subset \text{Cl}_X(U)$. The family of all semiopen sets in $(X, \tau)$ is denoted by $SO(X, \tau)$.

Definition 2.1. A topological space $(X, \tau)$ is said to be $S$-closed [11] if for every semiopen cover $\{U_\alpha | \alpha \in \mathcal{V}\}$ of $X$ there exists a finite subfamily $\mathcal{V}_0$ of $\mathcal{V}$ such that $X = \bigcup \{\text{Cl}_X(U_\alpha) | \alpha \in \mathcal{V}_0\}$.

A subset $S$ of $(X, \tau)$ is said to be $S$-closed if it is $S$-closed as the subspace of $(X, \tau)$. A subset $S$ of $(X, \tau)$ is said to be $S$-closed relative to $\tau$ [6] if for every cover $\{U_\alpha | \alpha \in \mathcal{V}\}$ of $S$ by semiopen sets of $(X, \tau)$ there exists a finite subfamily $\mathcal{V}_0$ of $\mathcal{V}$ such that $S \subset \bigcup \{\text{Cl}_X(U_\alpha) | \alpha \in \mathcal{V}_0\}$.

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DEFINITION 2.2. A topological space \((X, \tau)\) is said to be locally \(S\)-closed [7] if each point of \(X\) has an open neighborhood which is an \(S\)-closed subspace of \((X, \tau)\).

Every \(S\)-closed space is locally \(S\)-closed. However, the converse is not true in general because an infinite discrete space is locally \(S\)-closed but not \(S\)-closed. The following lemmas shown in [7] will be used in the sequel.

**Lemma 2.3.** Let \(A\) and \(B\) be subsets of a topological space \((X, \tau)\). If \(A\) is \(S\)-closed relative to \(\tau\) and \(B\) is regular open, then \(A \cap B\) is \(S\)-closed relative to \(\tau\).

**Lemma 2.4.** For a topological space \((X, \tau)\), the following are equivalent.
1. \((X, \tau)\) is locally \(S\)-closed.
2. For each \(x \in X\), there exists \(U \in \tau\) such that \(x \in U\) and \(U\) is \(S\)-closed relative to \(\tau\).
3. For each \(x \in X\), there exists \(U \in \tau\) such that \(x \in U\) and \(\text{Int}_X(\text{Cl}_X(U))\) is \(S\)-closed relative to \(\tau\).

3. The results.

**Definition 3.1.** A topological space \((X, \tau)\) is said to be weakly Hausdorff [10] if each point of \(X\) is an intersection of regular closed sets of \((X, \tau)\).

**Theorem 3.2.** If a topological space \((X, \tau)\) is locally \(S\)-closed and weakly Hausdorff, then it is extremally disconnected.

**Proof.** Assume that \((X, \tau)\) is not extremally disconnected. Then, there exists a regular open set \(G\) of \((X, \tau)\) such that \(\text{Cl}_X(G) - G \neq \emptyset\) and \(X - \text{Cl}_X(G) \neq \emptyset\). Let \(x \in \text{Cl}_X(G) - G\). By Lemma 2.4, there exists an open neighborhood \(V\) of \(x\) such that \(V\) is \(S\)-closed relative to \(\tau\). Put \(A = G \cap V\), then by Lemma 2.3, \(A\) is \(S\)-closed relative to \(\tau\). Since \((X, \tau)\) is weakly Hausdorff and \(x \notin A\), for each \(a \in A\) there exists a regular closed set \(F(a)\) such that \(x \notin F(a)\) and \(a \in F(a)\). Since \(F(a) \in \text{SO}(X, \tau)\), there exists a finite subset \(A_0\) of \(A\) such that \(A \subseteq \bigcup \{F(a) | a \in A_0\}\). We have \(A \subseteq V \cap \text{Cl}_X(G) \subseteq \text{Cl}_X(V \cap G) = \text{Cl}_X(A)\). Therefore, \(x \in \text{Cl}_X(A) \subseteq \bigcup \{F(a) | a \in A_0\}\). On the other hand, \(x \notin F(a)\) for any \(a \in A_0\) and hence \(x \notin \bigcup \{F(a) | a \in A_0\}\). This contradiction implies that \((X, \tau)\) is extremally disconnected.

**Corollary 3.3.** A weakly Hausdorff space \((X, \tau)\) is \(S\)-closed if and only if it is locally \(S\)-closed and quasi \(H\)-closed.

**Definition 3.4.** A topological space \((X, \tau)\) is said to be almost-regular [9] if for each \(x \in X\) and each regular closed set \(F\) not containing \(x\) there exist disjoint open sets \(U\) and \(V\) of \((X, \tau)\) such that \(x \in U\) and \(F \subseteq V\).

**Theorem 3.5.** If a topological space \((X, \tau)\) is locally \(S\)-closed and almost-regular, then it is extremally disconnected.

**Proof.** Assume that \((X, \tau)\) is not extremally disconnected. Then there exists a regular open set \(G\) of \((X, \tau)\) such that \(\text{Cl}_X(G) - G \neq \emptyset\) and \(X - \text{Cl}_X(G) \neq \emptyset\). Let \(x \in \text{Cl}_X(G) - G\). Then, by Lemma 2.4, there exists a regular open neighborhood \(O\) of \(x\) which is \(S\)-closed relative to \(\tau\). Put \(A = O \cap G\), then by Theorem 1.2 of [6]...
and Lemma 2.3, $A$ is an $S$-closed subspace of $(X, \tau)$. Let $\mathcal{B}(x)$ be the family of all neighborhoods at $x$ and $\mathcal{S} = \{ V \cap A | V \in \mathcal{B}(x) \}$. Then, $\mathcal{S}$ is a filter base on $A$ and hence, by Theorem 2 of [11], $S$-accumulates to some point $a \in A$. Since $(X, \tau)$ is almost-regular and $A$ is regular open, there exists $U \in \tau$ such that $a \in U \subset \text{Cl}_X(U) \subset A$ [9, Theorem 2.2]. Since $x \notin A$, $(X - \text{Cl}_X(U)) \cap A \in \mathcal{S}$ and $a \in U \in SO(A)$. Moreover, we have $[(X - \text{Cl}_X(U)) \cap A] \cap \text{Cl}_A(U) = \emptyset$ which contradicts that $\mathcal{S}$ $S$-accumulates to $a \in A$. This shows that $(X, \tau)$ is extremally disconnected.

**Corollary 3.6.** An almost-regular space $(X, \tau)$ is $S$-closed if and only if it is locally $S$-closed and quasi $H$-closed.

**Definition 3.7.** A locally $S$-closed space $(X, \tau)$ is said to be maximal locally $S$-closed if $\tau = \theta$ whenever a topological space $(X, \theta)$ is locally $S$-closed and $\theta$ is stronger than $\tau$.

A function $f : (X, \tau) \to (Y, \theta)$ is said to be irresolute [2] if $f^{-1}(V) \in SO(X, \tau)$ for every $V \in SO(Y, \theta)$.

**Lemma 3.8.** Let $f : (X, \tau) \to (Y, \theta)$ be an irresolute function. If $G \in \tau$ and $G$ is $S$-closed in $(X, \tau)$, then $f(G)$ is $S$-closed in $(Y, \theta)$.

**Proof.** Let $f_G : G \to f(G)$ be a function defined by $f_G(x) = f(x)$ for every $x \in G$, where $G$ (resp. $f(G)$) is the subspace of $(X, \tau)$ (resp. $(Y, \theta)$). We shall show that $f_G$ is irresolute. For any $V_0 \in SO(f(G))$, there exists $V \in SO(Y, \theta)$ such that $V_0 = V \cap f(G)$ [8, Theorem 3.2]. Since $f$ is irresolute and $G \in \tau$, $f^{-1}(V) \cap G \in SO(X, \tau)$ and hence, by Theorem 1 of [5], $f_G^{-1}(V_0) = f^{-1}(V) \cap G \in SO(G)$. This shows that $f_G$ is irresolute. Since $G$ is $S$-closed, it follows from Theorem 3.5 of [12] that $f_G(G) = f(G)$ is $S$-closed.

**Theorem 3.9.** If a topological space $(X, \tau)$ is maximal locally $S$-closed, then it is extremally disconnected.

**Proof.** Assume that $(X, \tau)$ is not extremally disconnected. Then, there exists a regular closed set $B$ of $(X, \tau)$ such that $B \notin \tau$. Put $\tau(B) = \{ U \cup (V \cap B) | U, V \in \tau \}$, then $\tau(B)$ is a topology on $X$ which is strictly stronger than $\tau$. We shall show that $(X, \tau(B))$ is locally $S$-closed. Let $i_{\tau} : (X, \tau) \to (X, \tau(B))$ be the identity function. Then, it is obvious that $i_{\tau}$ is open. For each $U \cup (V \cap B) \in \tau(B)$, $V \cap B \in SO(X, \tau)$ and hence $U \cup (V \cap B) \in SO(X, \tau)$. This shows that $i_{\tau}$ is semicontinuous. Therefore, $i_{\tau}$ is irresolute [5, Theorem 7]. Since $(X, \tau)$ is locally $S$-closed, for each $x \in X$ there exists an open neighborhood $U$ of $x$ in $(X, \tau)$ such that $U$ is $S$-closed. By Lemma 3.8, $i_{\tau}(U)$ is $S$-closed in $(X, \tau(B))$. Moreover, $i_{\tau}(U)$ is an open neighborhood of $x$ in $(X, \tau(B))$ because $i_{\tau}$ is open. This shows that $(X, \tau(B))$ is locally $S$-closed. This contradicts that $(X, \tau)$ is maximal locally $S$-closed. Therefore, $(X, \tau)$ is extremally disconnected.

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REFERENCES


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