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SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON A PROBLEM OF L. NACHBIN

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ABSTRACT. If B is an uncountable set then there is a function $r: B \times B \to \mathbf{R}_+$ for which there is no function $t: B \to \mathbf{R}_+$ such that

$$r(b_1, b_2) < t(b_1) \cdot t(b_2) \quad \text{for all } b_1, b_2 \in B.$$

THEOREM. *If B is an uncountable set then there exists a function $r: B \times B \to \mathbf{N}$ with the property that for every function $t: B \to \mathbf{N}$ there are $a, b \in B$ such that $r(a, b) \geqslant t(a)$ and $r(a, b) \geqslant t(b)$.*

Here $\mathbf{N}$ denotes the set of all positive integers. Note that the theorem fails if B is countable. Let $\mathbf{R}_+$ be the set of all nonnegative reals.

COROLLARY. *If B is an uncountable set then there is a function $r: B \times B \to \mathbf{R}_+$ for which there is no function $t: B \to \mathbf{R}_+$ such that*

$$r(b_1, b_2) < t(b_1) \cdot t(b_2) \quad \textit{for all } b_1, b_2 \in B.$$

PROOF. Replace the function r by its square r^2. $\square$

This corollary answers a question of Leopoldo Nachbin. In **[1]**, the corollary is proved for sets B whose size is at least the continuum and is used (in Lemma 19) to give an example of a discontinuous 2-homogeneous complex polynomial on a locally convex complex vector space (with basis B).

PROOF OF THEOREM. It suffices to prove the theorem for a set B of cardinality $\aleph_1$. Thus let us assume that B is the set of all countable ordinal numbers. Let $<$ be the well ordering of ordinal numbers.

If $a \in B$, let W_a be the set of all ordinal numbers that are smaller than a; each W_a is at most countable. For each $a \in B$, let r_a be some one-to-one mapping of W_a into $\mathbf{N}$. If $a, b \in B$, we define $r(a, b) = r_a(b)$ whenever $a > b$, and arbitrarily otherwise.

I shall show that the function $r: B \times B \to \mathbf{N}$ satisfies the statement of the theorem. Thus let t be an arbitrary function from B into $\mathbf{N}$. We want to find $a, b \in B$ such that $r(a, b) \geqslant t(a)$ and $r(a, b) \geqslant t(b)$.

Since B is uncountable,[2] there is an uncountable subset C of B such that t is constant on C, with value n. Let $a \in C$ be such that a has infinitely many

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predecessors in C. Since the values of $r(a, b)$ where b ranges over the predecessors of a in C are all distinct, there is one such b with $r(a, b) > n = t(b)$. $\square$

References

1. J. A. Barroso, M. C. Matos and L. Nachbin, *On holomorphy versus linearity in classifying locally convex spaces*, Infinite Dimensional Holomorphy and Applications (M. C. Matos, Ed.), North-Holland., Amsterdam, 1977, pp. 31–74.

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