GLOBALLY STABLE COMPLETE MINIMAL SURFACES IN $R^3$

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ABSTRACT. It is proved that a globally stable complete minimal surface in $R^3$ with finite total curvature is a plane.

The aim of this note is to give a proof of the following result.

THEOREM. Let $x: M \to R^3$ be a complete minimal immersion of a two-dimensional, orientable, $C^\infty$ manifold $M$. Assume that $x$ is globally stable and that the total curvature of $x$ is finite. Then $x(M) \subset R^3$ is a plane.

Here globally stable means that each compact subdomain $W \subset M$ is a nondegenerate minimum for the area function of the induced metric for all variations that keep the boundary $\partial W$ of $W$ fixed. The result is clearly false for minimal surfaces in $R^4$ and we do not know whether it remains true without the finiteness of the total curvature.1

Proof. Let $S^2 \subset R^3$ be the unit sphere with center at $(0, 0, 0)$. We first prove that given a finite number of points $p_1, \ldots, p_k \in S^2$, there exists a domain $D \subset S^2$ such that $\lambda_1(D) = 2$, where $\lambda_1$ is the first eigenvalue of the Laplacian $\Delta$ in $S^2$, and $D$ omits neighborhoods $U_i \subset S^2$ of $p_i$, $i = 1, \ldots, k$.

For each $p_i$, make a rotation of $S^2$ such that $p_i = (0, 0, 1)$ and define a function $u_i: S^2 \to R$ by the restriction to $S^2$ of

$$u_i(x, y, z) = 2 - z \log [(1 + z)/(1 - z)], \quad (x, y, z) \in R^3.$$ 

It is easily checked that $\Delta u_i + 2 u_i = 0$ (cf. [1, p. 519]) and that $\lim_{z \to \pm 1} u_i = -\infty$. Furthermore, $u_i$ is positive in a ring-shaped domain $D_i$ bounded by two parallels of $S^2$ and vanishes on the boundary $\partial D_i$. Thus $\lambda_1(D_i) = 2$. Now set $u = \sum_{i=1}^k u_i$ and define $D$ as a connected component of the set \{ $p \in S^2; u > 0$\}.

We claim that $D$ is not empty. To see that we use the fact that of all spherical domains with the same area, the spherical cap has the smallest eigenvalue [3]. Since $D_1 \cap D_2 \neq \emptyset$, a connected component $D_{12}$ of the set \{ $p \in S^2; u_1 + u_2 > 0$\} has eigenvalue 2. By the above, and the fact that the hemisphere has eigenvalue 2, we conclude that $A(D_{12}) > 2\pi$, where $A(\ )$ denotes the area of the enclosed domain.

By the same token, $A(D_i) > 2\pi$; hence $D_{12} \cap D_3 \neq \emptyset$. An easy induction now shows that $A(D) > 2\pi$, as we claimed. Thus $\lambda_1(D) = 2$ and, since $\lim_{p \to p_i} u = -\infty$, $D$ omits neighborhoods $U_i$ of $p_i$, as we wished to prove.

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1After the writing of this paper, this was shown to be true, independently, by Fischer-Colbrie, Schoen and do Carmo, Peng.

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We need to use some results of R. Osserman [2]. The finiteness of the total curvature implies that $M$ is conformally equivalent to a compact Riemann surface $R$ minus a finite number of points $q_1, \ldots, q_k$. Furthermore, the Gauss map $g: M \to S^2$ can be analytically extended to a map $\tilde{g}: R \to S^2$ and $\tilde{g}$ is either constant or a branched covering of $S^2$; thus, if $\tilde{g}$ is not constant, the (finitely many) points of $S^2$ that are omitted by $g(M)$ are all images of some $q_j$'s under $\tilde{g}$.

Now, assume that $x(M)$ is not a plane (i.e., $\tilde{g}$ is not constant) and let $p_i = \tilde{g}(q_i)$. By the previous construction, we can find a domain $D \subset S^2$ that omits neighborhoods $U_i$ of $p_i$ and such that $\lambda_1(D) = 2$. Let $\{q_1, \ldots, q_k, q_{k+1}, \ldots, q_n\} \subset R$ be the inverse image by $\tilde{g}$ of the set $\{p_1, \ldots, p_k\}$. Choose neighborhoods $V_j \subset R$ of $q_j, j = 1, \ldots, n$, such that each $\tilde{V}_j$ contains at most $q_j$ as a branch point for $\tilde{g}$, and $\tilde{g}(V_j)$ is contained properly in some $U_i$. Set $M \setminus \bigcup_j V_j = W$. Thus $\overline{W}$ is compact and, since $\tilde{g}$ is a branched covering, $g(W) \not\subset D$. It follows that $\lambda_1(g(W)) < 2$. From Theorem 2.7 in [1], we conclude that $W$ is unstable. This is a contradiction to the fact that $x$ is globally stable, and completes the proof.

REFERENCES


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