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TRACE POLYNOMIAL FOR TWO GENERATOR SUBGROUPS OF SL(2, C)

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ABSTRACT. If G is a group generated by two 2×2 matrices A and B having determinant $+1$, with entries from the complex field $\mathbf{C}$, it is known that the trace of any word in A and B, $W(A, B)$ is a polynomial with integral coefficients in the three variables: $x = \text{trace}(A), y = \text{trace}(B), z = \text{trace}(AB)$, defined as

$$\text{trace } W(A, B) = P(x, y, z),$$

where P is determined uniquely by the conjugacy class of $W(A, B)$.

The actual computation of this trace polynomial is not easily obtained. It is the purpose of this paper to derive an explicit formula for this trace polynomial, and to indicate some consequences of it.

1. Introduction. Fricke, in his work on automorphic functions [1] had shown the following:

Let G be a group generated by finitely many 2 by 2 matrices with entries from the field $\mathbf{C}$ of complex numbers and with determinant $+1$. Then the trace of any word in the generators is a polynomial with integral coefficients in finitely many variables which are the traces of the generators and of finitely many of their products. In particular, if G has two generators A and B, then the trace of any word $W(A, B)$ is a polynomial in x, y, z where

$$x = \text{trace } A, \qquad y = \text{trace } B, \qquad z = \text{trace } AB. \tag{1}$$

Horowitz [2] has shown that x, y, z can be considered as independent variables and that for every triplet of values x_0, y_0, z_0 there exists a pair of unimodular matrices A_0, B_0 such that trace $A_0 = x_0$, trace $B_0 = y_0$ and trace $A_0B_0 = z_0$. In addition, A_0, B_0 are, in general, i.e. whenever

$$x_0^2 + y_0^2 + z_0^2 - x_0y_0z_0 - 4 \neq 0, \tag{2}$$

uniquely determined up to conjugacy within the linear group SL(2, $\mathbf{C}$). Since the group generated by two matrices A, B is free of rank 2 unless x, y, z satisfy at least one of a countable number of algebraic equations with integral coefficients, it follows that every conjugacy class of an element $W(a, b)$ in a free group on free generators a, b determines uniquely a polynomial $P(x, y, z)$ with integral coefficients if we define P by

$$\text{trace } W(A, B) = P(x, y, z). \tag{3}$$

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But Horowitz also showed that P may not determine the conjugacy class of $W(a, b)$ uniquely, although, for a given P, there can exist only finitely many conjugacy classes represented by cyclically reduced words $W(a, b)$ such that (3) holds.

The proof of Fricke's theorem is based on the simple identities

$$\begin{gathered}\operatorname{trace} UV + \operatorname{trace} UV^{-1} = \operatorname{trace} U \cdot \operatorname{trace} V, \\ \operatorname{trace} U^{-1} = \operatorname{trace} U = \operatorname{trace} WUW^{-1}\end{gathered} \tag{4}$$

which is valid for any W and any pair U, V of elements of SL(2, C). However, (4) does not lend itself easily to the actual explicit computation of trace $W(A, B)$ for an arbitrary W. On the other hand, applications of the results of Fricke and Horowitz to other group theoretical problems, for instance those treated by Rhee [6], Whittemore [7], Lyndon and Ullman [4] or Magnus [5] show that such explicit formulas would be very useful. It is the purpose of the present paper to derive and discuss formulas of this type.

Let A and B be two 2×2 matrices with entries from the complex field C, having determinant $+1$.

Let $x = \operatorname{trace}(A), y = \operatorname{trace}(B), z = \operatorname{trace}(AB)$.

Throughout the discussion, the following properties of the trace of matrices U, V having determinant $+1$ will be used:

(1) $\operatorname{tr}(UV) = \operatorname{tr}(VU)$,

(2) $\operatorname{tr}(UV) = \operatorname{tr}(U) \cdot \operatorname{tr}(V) - \operatorname{tr}(UV^{-1})$,

(3) $\operatorname{tr}(U^{-1}) = \operatorname{tr}(U)$,

and if $u = \operatorname{tr}(U)$, then for any $n \in Z$, $\operatorname{tr}(U^n) = T_n(u)$, where $T_n(t)$ is the Chebyshev polynomial in the variable t, defined recursively as follows:

$$T_0(t) = 2, \quad T_1(t) = t,$$
$$T_n(t) = tT_{n-1}(t) - T_{n-2}(t), \quad T_n(2 \cos \varphi) = 2 \cos n\varphi.$$

We also need the polynomials $P_n(t)$ which satisfy the recurrence relations $P_n(t) = tP_{n-1}(t) - P_{n-2}(t)$, $n \in Z^+$, and the initial conditions $P_0(t) = 1$, $P_{-1}(t) = 0$. We have

$$P_n(2 \cos \varphi) = \sin(n + 1)\varphi / \sin \varphi.$$

The degree of $P_n(t)$ in t is n for $n \geqslant 2$ and $|n| - 2$ for $n < 1$. $P_{-1}(t)$ is identically zero and $P_n(t) = -P_{|n|-2}(t)$ for $n < 0$.

The polynomials $P_n(t)$ have as their generating function

$$P(t, u) = (u^2 - ut + 1)^{-1}$$

where u is an arbitrary parameter.

The following property of the zeros of the polynomials $P_n(t)$, $n \geqslant 2$, follows from examining their trigonometric representation, and is useful in establishing some of the results.

Let $n \in Z^+$, $n \geqslant 2$. Then $P_n(t)$ has n zeros, defined by

$$t(n, k) = 2 \cos(k\pi / (n + 1)), \qquad k = 1, 2, \ldots, n,$$

in the interval $[-1, 1]$, with largest zero occurring at $t(n, 1) = 2 \cos(\pi/(n + 1))$.

These polynomials are related to the Chebyshev polynomials $T_n(t)$ as follows:

$$T_n(t) = P_n(t) - P_{n-2}(t), \quad \text{for } n \geq 2.$$

The proofs of all results are omitted, since once the polynomials P_n have been introduced, the results follow easily by induction, and properties (1), (2), (3) of the trace of matrices with determinant $+1$, previously stated.

2. The main theorem.

THEOREM 1. *Let* $D = A^{u_1}B^{v_1} \cdots A^{u_n}B^{v_n}$, *where* u_i, $v_i \in Z - \{0\}$ *for* $i = 1, 2, \ldots, n$, *be a cyclically reduced n syllable word. Then*

$$\operatorname{tr}(D) = M_n(x, y)z^n + M_{n-1}(x, y)z^{n-1} + \cdots + M_0(x, y),$$

where the $M_i(x, y)$, $i = 0, 1, \ldots, n$, *are sums of products of the polynomials* $P_u(x)$, $P_v(y)$. *Moreover, if* d_i *represents the degree of* $M_i(x, y)$, *then*

$$\sum_{i=1}^{n} (|u_i| + |v_i|) - 2n \leq d_i \leq \sum_{i=1}^{n} (|u_i| + |v_i|).$$

In particular, $M_n(x, y) = \prod_{i=1}^{n} P_{u_i-1}(x)P_{v_i-1}(y)$, *and so* $d_n = \sum_{i=1}^{n}(|u_i| + |v_i|) - 2n$. *Thus, the degree of* $\operatorname{tr}(D)$ *in z equals the number of blocks* $A^{u_i}B^{v_i}$.

3. Consequence of the main theorem.

COROLLARY 1. *Cyclically reduced words* W_1 *and* W_2 *can have the same trace polynomial only if the absolute values of the exponents of the generators of a in* W_2 *arise from those in* W_1 *by a permutation, and the same must be true for the exponents of b.*

4. Generating function for traces of n syllable words. Let t_i, s_i be arbitrary scalar parameters, such that $0 < |t_i| < 1$, $0 < |s_i| < 1$, for $i = 1, 2, 3, \ldots, l$. Consider the following matrix product

$$\prod_{i=1}^{l} t_i A^{\varepsilon_i}(I - t_i A^{\varepsilon_i})^{-1} s_i B^{\eta_i}(I - s_i B^{\eta_i})^{-1}, \tag{1}$$

where ε_i, $\eta_i = \pm 1$ for $i = 1, 2, \ldots, l$; and I is the 2×2 identity matrix. By considering two different representations for this product based on the results

$$(I - tM^{\varepsilon})^{-1} = \sum_{j=0}^{\infty} t^j M^{\varepsilon j}, \qquad (I - tM^{\varepsilon})^{-1} = (I - tM^{-\varepsilon}) \sum_{j=0}^{\infty} P_j(u)t^j,$$

where M is any 2×2 matrix with $\det(M) = +1$, $u = \operatorname{tr}(M)$, $\varepsilon = \pm 1$; one obtains the following expression involving the trace of an n syllable word in A and B:

$$H(t_i, x, s_i, y)\operatorname{tr}\left[(A^{\varepsilon_1} - t_1 I)(B^{\eta_1} - s_1 I) \cdots (A^{\varepsilon_l} - t_l I)(B^{\eta_l} - s_l I)\right]$$
$$= \sum_{\substack{j_i, k_i = 0 \\ 1 \leq i \leq l}} t_1^{j_1+1} s_1^{k_1+1} \cdots t_l^{j_l+1} s_l^{k_l+1} \operatorname{tr}(A^{\varepsilon_1(j_1+1)} B^{\eta_1(k_1+1)} \cdots A^{\varepsilon_l(j_l+1)} B^{\eta_l(k_l+1)}), \tag{2}$$

where

$$H(t_i, x, s_i, y) = \sum_{\substack{j_i, k_i = 0 \\ 1 \leq i \leq l}}^{\infty} \prod_{i=1}^{l} \left[P_{j_i}(x)P_{k_i}(y)t_i^{j_i+1}s_i^{k_i+1}\right].$$

If one considers fixed ε_i, $\eta_i = \pm 1$; $i = 1, 2, 3, \ldots, l$; then from (2), one sees that $\operatorname{tr}(A^{\varepsilon_1 n_1} B^{\eta_1 m_1} \cdots A^{\varepsilon_l n_l} B^{\eta_l m_l})$, where n_i, $m_i = 1, 2, 3, \ldots$ will involve sums of products of the polynomials $P_n(x)$, $P_m(y)$ and the trace of the matrix product given by

$$\operatorname{tr}\left[\prod_{i=1}^{l} (A^{\varepsilon_i} - t_i I)(B^{\eta_i} - s_i I)\right].$$

From this one sees that for $\operatorname{tr}(A^{\varepsilon_1 n_1} B^{\eta_1 m_1} \cdots A^{\varepsilon_l n_l} B^{\eta_l m_l})$, where n_i, $m_i = 1, 2, 3, \ldots$ for $1 \leq i \leq l$, and ε_i, $\eta_i = \pm 1$ for $1 \leq i \leq l$ are fixed, the trace is determined by the choice of the ε_i, η_i. In fact, the signs of each pair (ε_i, η_i); $\varepsilon_i \eta_i > 0$ or $\varepsilon_i \eta_i < 0$, will determine the degree of the trace polynomial in x, y, z, and the signs and degrees of the coefficients of the variable z.

Thus, in any rearrangement of the exponents, the signs of the products $\varepsilon_i \eta_i$ cannot be altered. Moreover, from the development of the generating function for an n syllable word, it can be seen that one cannot interchange exponents of the t_i and s_i, since this would give rise to a different term in the power series.

Hence, the only rearrangement of the exponents appearing in $\operatorname{tr}(A^{\varepsilon_1 n_1} B^{\eta_1 m_1} \cdots A^{\varepsilon_l n_l} B^{\eta_l m_l})$, where $n_i, m_i \in Z^+$, $1 \leq i \leq l$, $\varepsilon_i, \eta_i = \pm 1$, $1 \leq i \leq l$, are fixed; which can occur without changing the trace, is one which can be done without altering the signs of the products $\varepsilon_i \eta_i$. This leads to the following:

THEOREM 2. *In words of syllable length n: $A^{\alpha_1} B^{\beta_1} \cdots A^{\alpha_n} B^{\beta_n}$ we can apply a permutation π to the α_ν without changing the traces if and only if this can be done in the case when all the α_ν (and all the β_ν) have value ± 1.*

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BIBLIOGRAPHY

1. R. Fricke and F. Klein, *Vorlesungen ueber die Theorie der automorphen Functionen*, Chapter 2, Teubner, Leipzig, 1897.

2. R. Horowitz, *Characters of free groups represented in the two-dimensional linear group*, Comm. Pure Appl. Math. **25** (1972), 635–649.

3. _____, *Induced automorphisms of Fricke characters of free groups*, Trans. Amer. Math. Soc. **208** (1975), 41–50.

4. R. C. Lyndon and J. L. Ullman, *Groups generated by two parabolic linear fractional transformations*, Canad. J. Math. **21** (1969), 1388–1403.

5. W. Magnus, *Two generator subgroups of PSL*(2, C), Nachrichten der Akademie der Wissenchaften in Gottingen Mathematisch-Physikalische Klasse, no. 7, Jahrgang, 1975, pp. 1–14.

6. R. Ree and N. S. Mendelsohn, *Free subgroups with a single defining relation*, Arch. Math. **19** (1974), 577–580.

7. Alice Whittemore, *On representations of the group of Listing's knot by subgroups of PSL*(2, C), Proc. Amer. Math. Soc. **40** (1973), 378–382.

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