A REMARK ON COSINE FAMILIES

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ABSTRACT. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family and $A$ its infinitesimal generator. Then the set $E = \{ x \in X : C(t)x \text{ is once continuously differentiable in } t \text{ on } \mathbb{R} \}$ of the Banach space $X$ is contained in the domain of $(-A)^{\alpha}$ for $0 < \alpha < 1/2$.

The purpose of this note is to prove for a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, defined on a Banach space $X$ and with infinitesimal generator $A$, that the set $E = \{ x \in X : C(t)x \text{ is once continuously differentiable in } t \text{ on } \mathbb{R} \}$ is contained in the set $D[(-A)^{\alpha}]$, $0 < \alpha < 1/2$. The set $D[(-A)^{\alpha}]$ is the domain of the $\alpha$ power of the operator $-A$.

A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators mapping the Banach space $X$ into itself is called a strongly continuous cosine family if and only if

\begin{align*}
C(t + s) + C(t - s) &= 2C(t)C(s) \text{ for all } s, t \in \mathbb{R}, \\
C(0) &= I, \\
C(t)x \text{ is continuous in } t \in \mathbb{R} \text{ for each fixed } x \in X.
\end{align*}

The associated sine family is given by $S(t)x = \int_0^t C(s)x \, ds$ for $x \in X$ and $t \in \mathbb{R}$. The linear operator $A : X \to X$ defined by $Ax = C''(0)x$ and with dense domain $D(A) = \{ x \in X : C(t)x \text{ is twice continuously differentiable in } t \text{ on } \mathbb{R} \}$ is called the infinitesimal generator of $C(t)$, $t \in \mathbb{R}$. For other properties of cosine families used in this paper see [2] or [9].

The following theorem appears in C. Travis and G. Webb [8] and [9]:

THEOREM 1. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with associated sine family $S(t)$, $t \in \mathbb{R}$, and infinitesimal generator $A$. The following statements are equivalent:

(i) there exists a closed linear operator $B$ on $X$ such that $B^2 = A$ and $B$ commutes with every operator in $B(X, X)$ which commutes with $A$; $S(t)$ maps $X$ into $D(B)$ for each $t \in \mathbb{R}$; $BS(t)x$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$;

(ii) $E = D(B)$, the domain of $B$.

The conditions stated in part (i) of Theorem 1 have also been considered by H. Fattorini in [2] and [3]. Fattorini has shown in [3] that every strongly continuous cosine family defined on the Banach space $L^p$, $1 < p < \infty$, satisfies condition (i).
However, there are cosine families which do not satisfy condition (i) as shown by J. Kisynski [4] and B. Nagy [6]. It remains open whether or not $E \subset D[(\lambda - A)^{1/2}]$ in general.

**Proposition.** Let $A$ be the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, such that there exists an $M > 0$ with $\|((\lambda - A)^{-1}) \| \leq M/\lambda$ for all $\lambda > 0$. Then the set $E$ is contained in $D[(\lambda - A)^{\alpha}]$ for $0 < \alpha < 1/2$.

**Proof.** It is known, V. Balakrishnan [1], that $-(\lambda - A)^{1/2}$ generates an analytic semigroup $T(t)$ with $T(t)X \subset D(A)$. Now if $x \in E$ then $S(t)x$ is twice continuously differentiable and the function

$$u(t) = \int_0^t T(t - v)S(v)x \, dv$$

solves the differential equation

$$u'(t) = - (\lambda - A)^{1/2}u(t) + S(t)x, \quad u(0) = 0.$$  

Following a change of variable in the integral, we have

$$u'(t) = \int_0^t T(v)C(t - v)x \, dv$$

and

$$u''(t) = \int_0^t T(v)AS(t - v)x \, dv + T(t)x$$

$$= Au(t) + T(t)x = -(\lambda - A)^{1/2}\int_0^t T(v)C(v)x \, dv + C(t)x$$

$$= - (\lambda - A)^{1/2}u'(t) + C(t)x, \quad t > 0.$$  

We can now write for $t > 0$

$$Au(t) + T(t)x = -(\lambda - A)^{1/2}\left[-(\lambda - A)^{1/2}u(t) + S(t)x\right] + C(t)x$$

$$= -Au(t) - (\lambda - A)^{1/2}S(t)x + C(t)x,$$

which implies that

$$C(t)x = 2Au(t) + T(t)x + (\lambda - A)^{1/2}S(t)x$$

for all $x \in E$ and $t > 0$.

Under the stated conditions the fractional power of the operator $(\lambda - A)^{1/2}$ exist and $\|((\lambda - A)^{1/2})^{\beta}T(t)\| < C/t^{\beta}$ for all $t > 0$, for some $C > 0$, and for $0 < \beta < 1$. (See the book by S. Krein [5] or the lecture notes by A. Pazy [7].)

Therefore since $\int_0^t ((\lambda - A)^{1/2})^{\beta}T(t - v)AS(v)x \, dv$ $(t > 0)$ exists for any $0 < \beta < 1$, we have that $Au(t) = \int_0^t T(t - v)AS(v)x \, dv$ is in $D[(\lambda - A)^{1/2}]$, $t > 0$, for all $0 < \beta < 1$. This is equivalent to $Au(t) \in D[(\lambda - A)^{\alpha}]$, $t < 0$, for all $0 < \alpha < 1/2$.

Appealing now to equation (*), the fact that $(\lambda - A)^{1/2}S(t)x \in D[(\lambda - A)^{1/2}]$, $T(t)x \in D(A)$, and $Au(t) \in D[(\lambda - A)^{\beta}]$ $(0 < \alpha < 1/2)$, for all $t > 0$, we have that $C(t)x \in D[(\lambda - A)^{\alpha}]$ for $t > 0$. The identity $x = 2C(t)C(t)x - C(2t)x$ gives the result, since $C(t), t \in \mathbb{R}$, leaves $D[(\lambda - A)^{\alpha}]$ invariant.
REMARK. The condition $\| (\lambda - A)^{-1} \| < M/\lambda$ for $\lambda > 0$ is not as restrictive as it may seem since by Theorem 2.7 of [9], $\| (\lambda^2 - A)^{-1} \| < M/\lambda (\lambda - w)$ ($\lambda > w$). If $A$ generates a strongly continuous cosine family then so does $A_b = A - b^2 I$ [2]. Thus if one chooses $b > w$, we have for all $\lambda > 0$

$$\| (\lambda^2 - A_b)^{-1} \| = \| (\lambda^2 + b^2 - A)^{-1} \| < \frac{M}{\lambda (\lambda^2 + b^2 - w)} < \frac{M'}{\lambda^2}.$$  

Substituting $\sqrt{\lambda}$ for $\lambda$ the condition of the proposition is satisfied by $A_b$. In this sense (after a suitable translation) every strongly continuous cosine family satisfies the above proposition.

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REFERENCES