SETS WITH FIXED POINT PROPERTY FOR ISOMETRIC MAPPINGS

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ABSTRACT. A subset $K$ of a Banach space $E$ is said to have the fixed point property for isometric mappings (f. p. p.) if there exists $z$ in $K$ such that $U(z) = z$ for each isometric mapping $U$ from $K$ onto $K$. We prove that any bounded closed subsets of $E$ with uniform relative normal structure have the f. p. p. We also prove that if $E$ is either $B(H)$ (bounded operators on a Hilbert space $H$) or $l^\infty(X)$ (bounded functions on $X$), then $E$ is finite dimensional if and only if each weak*-compact convex subset of $E$ has the f. p. p. This is also equivalent to the convex set of (normal) states on $E$ having the f. p. p.

1. Introduction. A mapping $U: K \rightarrow E$ of a set $K$ in a Banach space $E$ into $E$ is called isometric if $\|Ux - Uy\| = \|x - y\|$ for all $x, y \in K$. We say that a subset $K$ of $E$ has the fixed point property for isometric mappings, or simply f. p. p., if there exists $z$ in $K$ such that $U(z) = z$ for any isometric mapping $U$ of $K$ onto itself. A well-known result of Brodskii and Milman [1] asserts that if $K$ is a weakly compact convex subset of $E$ and $K$ has normal structure, then $K$ has the f. p. p. In particular, any compact convex subset of $E$ has the f. p. p. (see [2, Lemma 1]).

We consider in this paper bounded closed subsets of a Banach space $E$ with the f. p. p. We prove, among other things, that any bounded closed subsets of $E$ with uniform relative normal structure have the f. p. p. We also show that if $E$ is either the space of bounded complex valued functions on a nonempty set $X$ with the supremum norm (denoted by $l^\infty(X)$), or the algebra of bounded operators on a Hilbert space $H$ (denoted by $B(H)$), then $E$ is finite dimensional if and only if the set of states (or normal states) on $E$ has the f. p. p. This is also equivalent to each weak*-compact convex space of $E$ has the f. p. p. Our proofs depend on several well-known properties of discrete amenable groups.

2. Preliminaries and some notations. If $E$ is the Banach space $l^\infty(X)$ or $B(H)$ as defined in §1, then $E$ has a unique predual $E_*$ (which is $l_1(X)$ in case $E = l^\infty(X)$, see [11] for details). A linear functional $\phi$ on $E$ is called a state if $\phi$ is positive and $\|\phi\| = 1$; $\phi$ is normal if $\phi \in E_*$. If $G$ is a group, then $G$ is amenable if there exists a state $\phi$ on $l^\infty(G)$ such that $\phi(l_a f) = \phi(f)$ for each $a \in G$, where $l_a f(x) = f(ax)$, $x \in G$. As is well known, any group containing the free group on two generators is not amenable (see [8, p. 236]).

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If $E$ is a Banach space, then the closed ball centre at $x \in E$ and radius $r > 0$ will be denoted by $B[x, r]$. We list below some subsets of $E$ that are known to have the fixed point property:

1. Any compact convex sets in $E$.
2. Any closed ball $B = B[x_0, r]$. In this case $x_0$ is a common fixed point for any isometric mapping $U$ from $B$ onto $B$. Indeed, if $U(x_0) = x$, $x \neq x_0$, define $y = x_0 - \frac{t}{\|x_0 - x\|}(x_0 - x)$, where $t = r/\|x_0 - x\|$, then $y \in B$ and $\|x - y\| = \|x - x_0\| + r > r$. However, if $w \in B$ is such that $U(w) = y$, then $\|x - y\| = \|U(x_0) - U(w)\| = \|x_0 - w\| < r$, which is impossible.
3. Any weakly compact convex set $A^\ast$ in $E$ if $E$ is strictly convex. In this case, any isometric mapping from $K$ onto $K$ is necessarily affine [3, Proposition 2]. Apply now the Ryll-Nardzewski fixed point theorem [7, p. 98].

It is not known whether there exist a weakly compact convex subset $A^\ast$ of a Banach space and a fixed point free isometry of $K$ onto $K$ (see [4] for a discussion of this open problem).

3. The main results. A closed subset $K$ of a Banach space $E$ is said to have uniform relative normal structure if there exists $0 < c < 1$ such that for any nonvoid bounded closed subset $M$ in $K$, there exist $z_M$ in $K$ such that

(i) $\|x - z_M\| \leq c\delta(M)$ for each $x \in M$,
(ii) if $y \in K$ such that $\|x - y\| \leq c\delta(M)$ for all $x \in M$, then
$$\|z_M - y\| \leq c\delta(M).$$

(Here $\delta(M)$ denotes the diameter of $M$.) The notion of uniform relative normal structure was introduced recently by P. M. Soardi [11]. He proved, among other things, that if $K$ is a nonempty weak*-closed set with uniform relative normal structure, and $T: K \to K$ is nonexpansive and leaves invariant a weak*-compact subset $M$ in $K$, then $K$ contains a fixed point for $T$. Examples of sets with uniform relative normal structure include any closed balls in $L^\infty(X, S, \mu)$ of a $\sigma$-finite measure space $(X, S, \mu)$ (see [11, p. 28]).

**Theorem 1.** Let $E$ be a Banach space and let $K$ be a bounded closed nonempty subset of $E$ with uniform relative normal structure, then $K$ has the f. p. p.

**Proof.** Let $\mathcal{U}$ denote the group of isometric mappings from $K$ onto $K$ and let $0 < c < 1$ be a real number satisfying conditions (i) and (ii) for uniform relative normality of $K$. Put $A_0 = K$, and let $r = \delta(K)$. If $r = 0$, the conclusion is trivial. Otherwise define
$$A_1 = \{ k \in A_0; A_0 \subseteq B[k, cr] \}.$$
Then $A_1$ is a closed nonempty subset of $K$ since $A_1$ is the intersection of $A_0$ with closed balls and $z_{A_0} \subseteq A_1$ by (i). Also $U(A_1) = A_1$ for each $U \in \mathcal{U}$. Indeed, if $h \in A_1$ and $x \in A_0$, let $y \in A_0$ such that $U(y) = x$. Then
$$\| U(h) - x\| = \| U(h) - U(y)\| = \| h - y\| \leq cr.$$
Hence \( U(H_0) \subseteq H_0 \). Equality follows by replacing \( U \) with \( U^{-1} \). Moreover \( \delta(A_1) < c_r \). Hence the set

\[
H_1 = \{ k \in K; A_1 \subseteq B[k, c^2r] \}
\]
is again closed, nonempty and \( U(H_1) = H_1 \) for each \( U \in \mathfrak{U} \). Define

\[
A_2 = \{ h \in H_1; H_1 \subseteq B[h, c^2r] \}.
\]

Then \( A_2 \) is closed, nonempty (since \( z_{A_1} \in A_2 \) by condition (ii)) \( \delta(A_2) < c^2r \) and \( U(A_2) = A_2 \) for each \( U \in \mathfrak{U} \). Repeating this process, we have defined a sequence of nonempty closed sets \( \{A_n\}_{n=1}^\infty \) in \( K \) with the following properties:

(i) \( U(A_n) = A_n \) for each \( U \in \mathfrak{U} \),
(ii) \( \delta(A_n) < c^n r \),
(iii) \( \|x - y\| < c^n r \) if \( x \in A_{n-1} \) and \( y \in A_n \).

For each \( n = 1, 2, \ldots \), pick \( z_n \in A_n \). Then, as readily checked, \( \{z_n\} \) is Cauchy, and \( U(z) = z \) for each \( U \in \mathfrak{U} \) if \( z \) is the limit point of \( \{z_n\} \) in \( K \).

**Theorem 2.** Let \( H \) be a Hilbert space. Then the following are equivalent:

(a) \( H \) is finite dimensional.
(b) The set of states on \( \mathfrak{B}(H) \) has the f. p. p.
(c) The set of normal states on \( \mathfrak{B}(H) \) has the f. p. p.
(d) Each weak*-compact convex subset of \( \mathfrak{B}(H) \) has the f. p. p.

**Proof.** If \( H \) is finite dimensional, then each of the sets in (b), (c) and (d) is compact and convex, and hence has the f. p. p.

Let \( \mathfrak{U}(H) \) denote the group of unitary elements in \( \mathfrak{B}(H) \). For each \( u \in \mathfrak{U}(H) \), define a weak*-weak*continuous isometric linear map from \( \mathfrak{B}(H) \) onto \( \mathfrak{B}(H) \) by \( \tau_u(x) = u^*xu, x \in \mathfrak{B}(H) \). If \( H \) is infinite dimensional, write \( H = l_2(G) \) where \( G \) is a free group, and \(|G| = \) the cardinality of a complete orthonormal basis for \( H \). For each \( g \in G \), the operator \( l_g: l_2(G) \rightarrow l_2(G) \) defined by \( l_g h(t) = h(gt), t \in G \), is in \( \mathfrak{B}(H) \).

Note that each of the sets in (b) and (c) is invariant under each \( \tau_u^*, u \in \mathfrak{U}(H) \). Hence if (b) or (c) holds, then there exists a state \( \phi \) on \( \mathfrak{B}(H) \) such that \( \tau_u^*\phi = \phi \) for each \( u \in \mathfrak{U}(H) \). For each \( f \in l_\omega(G) \), define \( x_f: l_2(G) \rightarrow l_2(G) \) by \( x_f(h) = f \cdot h \) (pointwise multiplication), \( h \in l_2(G) \). Then \( \|x_f\| < \|f\| \). Define \( m \in l_\omega(G)^* \) by

\[
m(f) = \phi(x_f), \quad f \in l_\omega(G).
\]

Then, as readily checked, \( m \) is a state on \( l_\omega(G) \). Also, if \( g \in G, f \in l_\omega(G) \) then \( l_g x_f l_{g^{-1}} = x_k \) where \( k = l_g f \). Hence

\[
m(f) = \phi(x_f) = \phi(l_g x_f l_{g^{-1}}) = m(l_g f),
\]

i.e. \( G \) is amenable, which is impossible. Hence \( H \) is finite dimensional.

Finally if (d) holds, and \( H \) is infinite dimensional, let \( VN(G) \) denote the weak*-closure of the linear span of \( \{l_g; g \in G\} \) in \( \mathfrak{B}(H) \). For each \( x \in \mathfrak{B}(H) \), let \( K(x) \) be the weak*-closure of the convex hull of \( \{l_g x l_g; g \in G\} \). Then \( K(x) \) is weak*-compact and invariant under each \( \tau_u \), when \( u = l_g, g \in G \). Hence there exists \( z \in K(x) \) such that \( l_g z l_g = z \) for each \( g \in G \). In particular, \( z \) is in the
commutant of $VN(G)$. By [10, Proposition 4.4.21], $G$ must be amenable, which is impossible.

**Theorem 3.** Let $X$ be a nonempty set. Then the following are equivalent:

(a) $X$ is finite.
(b) The set of states on $l_{\infty}(X)$ has the f. p. p.
(c) The set of normal states on $l_{\infty}(X)$ has the f. p. p.
(d) Each weak*-compact convex subset of $l_{\infty}(X)$ has the f. p. p.

**Proof.** If $X$ is finite, then $l_{\infty}(X)$ and $l_{\infty}(X)^*$ are finite dimensional, and hence (b), (c) and (d) hold.

If $X$ is infinite, then we may regard $X$ as a free group $G$ on $|X|$-generators. Also, the set of states (or normal states) on $l_{\infty}(G)$ is invariant under each of the isometrics $l^*_g$, $g \in G$. Hence if (b) or (c) holds, then there exists a state $\phi$ on $l_{\infty}(G)$ such that $l^*_g\phi = \phi$ for all $g \in G$, which is impossible. Finally, if (d) holds, then for each $f \in l_{\infty}(G)$, define $(r^g_f)(x) = f(xg)$ for each $x, a \in G$. Let $K(f)$ be the weak*-closed convex hull of $\{r^g_f; g \in G\}$. Then $K(f)$ is also weak*-compact. Hence by (d), there exists $h \in K(f)$, such that $r^g_h = h$ for all $g \in G$. Necessarily $h$ is a constant. By a result of Mitchell [9, Corollary 2], $G$ is amenable.

**Theorem 4.** Let $X$ be a nonempty set. Then the following are equivalent:

(a) $X$ has one element.
(b) The set of nonzero multiplicative linear functionals on $l_{\infty}(X)$ has the f. p. p.
(c) Each weak*-compact subset of $l_{\infty}(X)$ has the f. p. p.

**Proof.** That (a) implies (b) and (c) is trivial. To prove (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a), regard $X$ as a group. Then the set of nonzero multiplicative linear functionals on $l_{\infty}(G)$ is invariant under each of the isometric mappings $l^*_g$, $g \in G$. Now if (b) holds, then there exists a nonzero multiplicative linear functional $\phi$ on $l_{\infty}(G)$ such that $l^*_g\phi = \phi$ for all $g \in G$, which implies $G$ is trivial by [5, Corollary 3]. If (c) holds, then an argument similar to that of Theorem 3 shows that for each $f$ in $l_{\infty}(G)$, the weak*-closure of $\{r^g_f; g \in G\}$ contains a constant, which again implies that $G$ is trivial by [6, Theorem 1] and [5, Corollary 3].

**Remark.** Let $E$ be a dual Banach space, then each closed ball in $E$ is weak*-compact convex and has the f. p. p. Also, if $E$ is finite dimensional, then each weak*-compact convex subset of $E$ has the f. p. p. However, the converse is false unless $E = l_{\infty}(X)$, or $E = \mathcal{B}(H)$ by Theorems 2 and 3. (For example, take $E$ to be an infinite dimensional Hilbert space.)

**References**


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