IDEMPOTENT MULTIPLIERS ON SPACES OF CONTINUOUS FUNCTIONS WITH
\(p\)-SUMMABLE FOURIER TRANSFORMS

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Abstract. Let \(G\) denote a compact abelian group, and \(A^p\) the space of functions continuous on \(G\) and having \(p\)-summable Fourier transforms. The idempotent multipliers from \(A^p\) to \(A^q\) are characterised for \(p, q \in [1, 2]\).

Throughout \(G\) will denote a compact Hausdorff abelian group with character group \(\Gamma\). Given \(p \in [1, 2]\) we shall write \(A^p\) for the Banach space of functions continuous on \(G\) with \(p\)-summable Fourier transforms, and normed by \(\|f\| = \|f\|_\infty + \|\hat{f}\|_p\). Note that we can identify \(A^2\) with \(C\) and \(A^1\) with \(A\), the spaces of functions on \(G\) that are continuous and have absolutely convergent Fourier series respectively. An application of the Hahn-Banach theorem shows that the dual \((A^p)'\) of \(A^p\) can be identified with \(M + F\ell^p'\), where \(M\) is the space of Radon measures on \(G\) and \(F\ell^p'\) is the space of pseudomeasures on \(G\) with Fourier transforms in \(l^p'\). Here \(p'\) is the exponent conjugate to \(p\), that is, \(p' = p/(p - 1)\) with the usual convention if \(p = 1\). The duality is expressed by

\[ h(f) = \mu*f(0) + \sigma*f(0), \quad f \in A^p, \]

where \(h \in (A^p)'\), \(\mu \in M\) and \(\sigma \in F\ell^p'\).

We shall write \((A^p, A^q)\) for the set of pseudomeasures that are multipliers from \(A^p\) to \(A^q\); that is, \(\sigma \in (A^p, A^q)\) if and only if for every \(f \in A^p\) there exists \(g \in A^q\) with \(\sigma*f = g\). Using the above characterisation of the dual of \(A^p\) the following result can be proved.

Lemma 1. Given \(p \in [1, 2]\) and \(q \in [p, 2]\),


The \(A^p\) spaces and their multiplier spaces have been considered previously; see [1], [2], [5] and [6]. Here we are interested in characterising the set of multipliers \(\phi \in (A^p, A^q)\) that are idempotent; that is, satisfy \(\phi * \phi = \phi\). We shall write \((A^p, A^q)_e\) for the set of all idempotent elements of \((A^p, A^q)\). In addition, we shall denote by \(J\) the set of all idempotent measures on \(G\), by \(\xi_\Xi\) the characteristic function of the set \(\Xi\), and by \(|\Xi|\) the cardinality of \(\Xi\) if \(\Xi\) is finite.

By considering Fourier transforms we see that to each \(\phi \in (A^p, A^q)\), there corresponds unique \(\Xi \subseteq \Gamma\) for which \(\hat{\phi} = \xi_\Xi\). In the case \(p = 1, q \in [1, 2]\), it is easy to see (using Lemma 1) that

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(A, A^q)_f = (A, A)_f = \{\phi: \hat{\phi} = \xi_\Xi \text{ for some } \Xi \subseteq \Gamma\}.

For other values of p, q, p < q, the idempotent multipliers are precisely the idempotent measures.


**Proof.** The first equality follows directly from Lemma 1. As for the second, consider $\phi \in (A^p, A^p)_f$. By Lemma 1 we can write $\phi = \mu - \sigma$, where $\mu \in M$ and $\sigma \in l^p$. Choose a positive integer $n$ satisfying $2n < p' < 2(n + 1)$. Then $\sigma^{n+1} \in l^2$ and hence $\sigma^r \in L^2$ for all $r > n + 1$, where $\sigma^r$ denotes the $r$-fold convolution of $\sigma$ with itself. Now

$$
\mu = \phi + \sigma = \sum_{r=0}^{\infty} \left(\frac{s}{r}\right)^{p-r} \phi^{p-r} \ast \sigma^r = \sum_{r=0}^{n-1} \left(\frac{s}{r}\right)^{p-r} \phi \ast \sigma^r + \sigma^n
$$

(recall that $\phi^k = \phi$ for all $k > 1$) and consideration of this expression for $s = n + 1, n + 2, \ldots, 2n + 1$ gives

$$
\begin{bmatrix}
1 & (n + 1) & \ldots & (n + 1) \\
1 & (n + 2) & \ldots & (n + 2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & (2n + 1) & \ldots & (2n + 1)
\end{bmatrix}
\begin{bmatrix}
\phi \\
\phi \ast \sigma \\
\vdots \\
\phi \ast \sigma^n
\end{bmatrix}
= 
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_{n+1}
\end{bmatrix}
$$

where $\mu_i \in M$, $i = 1, 2, \ldots, n + 1$. The coefficient matrix is nonsingular; indeed using the identity $(t^{s+1} - s^n) = (t^n - s^n)$, $t > s$, it can be reduced to the upper triangular matrix $(a_{ij})$, where $a_{ij} = (t^{s+1} - s^n)_{i,j}$ for $i < j$. It follows that $\phi \in M$ and hence $\phi \in J$, thus showing that $(A^p, A^p)_f \subseteq J$. The reverse inclusion is clear, completing the proof of the theorem. □

For $p \in (1, 2]$, $q \in [1, p)$ the only idempotent multipliers $\phi$ from $A^p$ to $A^q$ are those given by $\phi = |s, \text{ where } s \text{ is finite. To prove this we require some preliminary results.}

Let $\mathcal{K}$ denote the family of all cosets in $\Gamma$ and $\mathcal{B}$ the Boolean ring generated by $\mathcal{K}$. Every $\Xi \in \mathcal{B}$ can be written as a finite union of sets of the form $\Xi = \cap_{i=1}^{n} \Lambda_i \cap \cap_{j=n+1}^{r} \Lambda_j$, where $\Lambda_i \in \mathcal{K}$ and $\Lambda_j$ denotes the complement of $\Lambda_i$. If $\Xi$ is infinite then so must be at least one such set $\Xi$.

**Lemma 2.** Given an infinite set $\Pi$ of the above type there exists a finite set $\mathcal{T}$ with the property that $\Pi \cup \mathcal{T}$ contains an infinite coset.

**Proof.** First we show that a finite intersection of cosets is either empty or itself a coset. Indeed consider $\Lambda_1, \Lambda_2 \in \mathcal{K}$ such that $\Lambda_1 \cap \Lambda_2$ is nonempty. Then $\Lambda_1 = \eta_1 + \Omega_1, \Lambda_2 = \eta_2 + \Omega_2$, where $\Omega_1, \Omega_2$ are subgroups of $\Gamma$, and there exist $\chi_1 \in \Omega_1, \chi_2 \in \Omega_2$ such that $\eta_1 + \chi_1 = \eta_2 + \chi_2$. Hence

$$
\Lambda_1 \cap \Lambda_2 = (\eta_2 + \chi_2 + \Omega_1) \cap (\eta_2 + \chi_2 + \Omega_2) = \eta_2 + \chi_2 + \Omega_1 \cap \Omega_2 \in \mathcal{K}.
$$
Thus a nonempty intersection of two cosets is itself a coset and the result extends to arbitrary finite intersections by induction.

Applying this to (the nonempty set) \( \Pi \) we see that

\[
\Pi = \bigcap_{i=1}^{n} \Lambda_i \cap \Lambda,
\]

By relabelling if necessary we can assume that \( \Pi \cup T \) contains the infinite set \( \bigcap_{i=1}^{n} \Lambda_i \cap \Lambda \), where \( T \) is finite and each \( \Lambda_i \) is infinite. Now \( \Lambda_i = \eta_i + \Omega_i \), \( \Lambda = \eta + \Omega \) for some subgroups \( \Omega_i \), \( \Omega \) of \( \Gamma \). We consider the following cases:

(1) \( \bigcap_{i=1}^{n} \Omega_i \cap \Omega \) is infinite. Then \( \Pi \cup T \) contains \( \bigcap_{i=1}^{n} \Lambda_i \cap \Lambda \) which, for suitable cosets \( \Lambda_i \) of \( \Omega_i \), is nonempty; just use the fact that each \( \Lambda_i \) is a union of cosets of \( \Omega_i \). By the argument used in the first part of the proof of the lemma, \( \bigcap_{i=1}^{n} \Lambda_i \cap \Lambda \) is an infinite coset in \( \Gamma \).

(2) \( \bigcap_{i=1}^{n} \Omega_i \cap \Omega \) is finite, \( \bigcap_{i \in I} \Omega_i \cap \Omega \) is infinite for some \( I \subseteq \{1, 2, \ldots, n\} \) with \( |I| = n - 1 \). Then there are cosets \( \Lambda'' \) of \( \Omega_i \) such that \( \Pi \cup T \) contains the nonempty set \( \bigcap_{i \in I} \Lambda'' \cap \Lambda \cap \Lambda' \), where \( \{j\} = \{1, 2, \ldots, n\} \setminus I \), and \( T' = \bigcap_{i \in I} \Lambda'' \cap \Lambda \cap \Lambda_j \) is (empty or) finite. Thus \( \Pi \cup T \cup T' = \bigcap_{i \in I} \Lambda'' \cap \Lambda \) an infinite coset.

(3) \( \bigcap_{i=1}^{n} \Omega_i \cap \Omega \) is finite for all \( I \subseteq \{1, 2, \ldots, n\} \) with \( |I| = n - 1 \). In this case we return to (2) above and repeat the process.

After \( n - 1 \) steps we have that \( \Omega_i \cap \Omega \) is finite for each \( i \in \{1, 2, \ldots, n\} \) and

\[
\Pi \cup T \cup \bigcap_{i=1}^{n} (\Lambda_i \cap \Lambda) \supset \left( \bigcap_{i=1}^{n} \Lambda_i \cap \Lambda \right) \cup \left( \bigcup_{i=1}^{n} \Lambda_i \cap \Lambda \right) = \Lambda
\]

is the required infinite coset. \( \Box \)

**Theorem 2.** Let \( \Xi \in \mathbb{R} \) be infinite and \( p \in (1, 2] \). Then there exists a continuous function \( f \) with \( \text{supp} f \subseteq \Xi \) and \( \hat{f} \in L^p \setminus L^q \) for all \( q \in [1, p) \).

**Proof.** By Lemma 2 we have the existence of a finite set \( T \) such that \( \Xi \cup T \) contains an infinite coset \( \Lambda \). We shall construct \( f \) as in the statement of the theorem with \( \text{supp} \hat{f} \subseteq \Lambda \), and then the result will follow. There are two cases:

(a) The subgroup \( \Omega = -\eta + \Lambda \) has an element \( \chi \) of infinite order. Now construct Rudin-Shapiro polynomials \( f_n \) in the usual way ([7, (37.19)]) so that \( \hat{f}_n \) takes values in \( \{-1, 0, 1\} \),

\[
\text{supp} \hat{f}_n = \{0, \chi, \ldots, (2^n - 1)\chi\} \quad \text{and} \quad \|f_n\|_{\infty} < 2^{(n+1)/2}.
\]

Choose \( \chi_n \in \Omega \) so that the spectra of the functions \( \chi_n f_n \) are pairwise disjoint and put \( f = \sum_{n=0}^{\infty} (n + 1)^{-2^{-n/p}} \eta \chi_n f_n \). Then

\[
\sum_{n=0}^{\infty} (n + 1)^{-2^{-n/p}} \|\eta \chi_n f_n\|_{\infty} < \infty,
\]

so that \( f \in C \). Clearly \( \text{supp} \hat{f} \subseteq \Lambda \). That \( f \in A^p \) follows from the estimate

\[
\|\hat{f}\|_p < \sum_{n=0}^{\infty} (n + 1)^{-2^{-n/p} 2^n/p} < \infty,
\]

whereas \( \|\hat{f}\|_q = \sum_{n=0}^{\infty} (n + 1)^{-2^{-n/p} q} 2^n \) shows that \( f \notin A^q \) for any \( q < p \).
(b) $-\eta + \Lambda = \Omega$ is a torsion subgroup of $\Gamma$, in which case there is a strictly increasing sequence $(\Omega_n)$ of finite subgroups of $\Omega$. For each $n$ construct $f_n$ (as in [4]) with the properties $\|f_n\|_1 = 1$ on $\Omega_n$, supp $f_n = \Omega_n$ and $\|f_n\|_\infty = |\Omega_n|^{1/2}$. As in (a) choose $\chi_n \in \Omega$ so that the spectra of the functions $\chi_n f_n$ are pairwise disjoint and put $f = \sum_{n=0}^{\infty} (n + 1)^{-2^{-n/p}} \eta \chi_n f_n$.

**Theorem 3.** Let $p \in (1, 2]$, $q \in [1, p)$ be given. Then $\phi \in (A^p, A^q)_I$ if and only if $\hat{\phi} = \xi_\Xi$ for some finite set $\Xi \subseteq \Gamma$.

**Proof.** Sufficiency is obvious. As for the other direction, consider $\phi \in (A^p, A^q)_I$. Using $(A^p, A^q)_I \subseteq (A^p, A^p)_I$ we see from Theorem 1 that $\phi \in J$ and hence, by [3, Theorem 3], $\hat{\phi} = \xi_\Xi$ for some set $\Xi$ belonging to the coset ring $\mathcal{R}$ of $\Gamma$. Suppose now that $\Xi$ is infinite. By Theorem 2 there exists $f \in C$ with supp $\hat{f} \subseteq \Xi$ and $\hat{f} \in l^p \setminus l^q$. This contradicts the assumption that $\hat{\phi} \in C\setminus l^q$.

**Corollary.** Suppose $G$ is infinite and let $p \in (1, 2]$, $q \in [1, p)$ be given. Then $(A^p, A^q)_I \subseteq (A^p, A^p)_I$.

**Proof.** That this inclusion is strict follows from Theorem 3 together with the fact that $\delta_0$ (the Dirac measure at 0) belongs to $(A^p, A^p)_I$.

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**References**


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