A CURVATURE NORMAL FORM FOR 4-DIMENSIONAL KÄHLER MANIFOLDS

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ABSTRACT. A curvature operator R is said to possess a *normal form* relative to some space of curvature operators $\mathcal{P}$ if R is determined uniquely in $\mathcal{P}$ by the critical points and critical values of the associated sectional curvature function. It is shown that any curvature operator of Kähler type in real dimension 4 with positive-definite Ricci curvature has a normal form relative to the space of all Kähler operators.

In [2] the author showed that a generic set of curvature operators of Kähler type with positive-definite Ricci tensor in real dimensions 4 and 6 possess *normal forms*, analogous to the diagonalization of symmetric operators but related to the sectional curvature. This concept arose out of work of Singer and Thorpe [3]; their normal form for Einstein 4-manifolds simplified greatly many classical integral expressions and showed plainly the relationship between curvature properties and characteristic numbers for these manifolds.

In this note, the author hopes to rectify, in real dimension 4, the major drawbacks of the normal form theorem proven in [2], which were the implicit nature of the normal form itself and the involved topological methods used in its proof.

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1. Definitions and statement of the theorem. A *curvature operator* $R \in \mathcal{R}(V)$ on a finite-dimensional inner product space V is a symmetric operator on $\Lambda^2(V)$. R is called *proper* if it satisfies the standard first Bianchi identity. If V is a hermitian complex vector space with complex structure automorphism $J: V \to V$, define $J \in \mathcal{R}(V)$ by $J(v \wedge w) = Jv \wedge Jw$. $R \in \mathcal{R}(V)$ is called *Kähler*, denoted by $R \in \mathcal{K}(V)$, if $RJ = JR = R$.

The Grassmannian $G(2, V)$ of oriented 2-planes in V is identified with $\{\xi \in \Lambda^2(V) | \, |\xi| = 1, \xi \wedge \xi = 0\}$ [3]. The *sectional curvature* $r_R: G(2, V) \to \mathbf{R}$ of $R \in \mathcal{R}(V)$ is then defined by $r_R(P) = \langle RP, P \rangle$.

Let $\mathcal{P} \subset \mathcal{R}(V)$ be a subspace. A proper $R \in \mathcal{P}$ *has a normal form relative to* $\mathcal{P}$ if there is a set $\{P_i\}$ of critical points of the sectional curvature r_R so that, if

$R' \in \mathcal{P}$ is proper and $r_{R'}$ has critical points at $\{P_i\}$ with $r_{R'}(P_i) = r_R(P_i)$, then $R' = R$.

THEOREM 1.1. *Let $V \simeq \mathbf{C}^2$. Then each proper $R \in \mathcal{K}(V)$ with positive-definite Ricci tensor has a normal form relative to $\mathcal{K}(V)$.*

2. Properties of $\mathcal{K}(V)$. Making the standard identification of $o(V) \simeq \Lambda^2(V)$, the subspace $u(V) \subset o(V)$ of skew-hermitian operators is the set of all J-invariant elements of $\Lambda^2(V)$. Define $I \in u(V)$ by identifying the operator J with an element I of $\Lambda^2(V)$. In a unitary basis $\{v_1, v_{1^*} = Jv_1, \ldots, v_n, v_{n^*}\}$, $I = \Sigma_i v_i \wedge v_{i^*}$.

In [2] the following useful formulas are derived; their proofs are simple calculations.

PROPOSITION 2.1. *Let $\rho(R): V \to V$ be the Ricci tensor of R. Then if R is proper*
(1) $\langle \rho(R)v, w \rangle = \langle R(I), v \wedge Jw \rangle$;
(2) *$\rho(R)$ is positive-definite if and only if $\langle R(I), v \wedge Jv \rangle > 0$ for each $v \in V$.*

A plane $P \in G(2, V)$ will be called *holomorphic* if $JP = P$. The set of all holomorphic planes is clearly 2 copies ($+$ and $-$) of $\mathbf{C}P(V)$.

PROPOSITION 2.2. (1) *$P \in \pm\mathbf{C}P(V)$ is a critical point of r_R if and only if P is a critical point of $r_{R|\pm\mathbf{C}P(V)}$. In the case $V \simeq \mathbf{C}^2$, P is a holomorphic critical point of r_R if and only if $RP = AP + B' * P$, where $A = r_R(P)$ and $*$ is the Hodge star operator.*
(2) *Let $V \simeq \mathbf{C}^2$. $Q \in G(2, V) - (\pm\mathbf{C}P(V))$ is a critical point of r_R if and only if $RQ = A(Q + JQ - \langle Q, I \rangle I)$, where $A = r_R(Q)$. Thus $\langle RQ, I \rangle = 0$.*

COROLLARY 2.3. *For $V \simeq \mathbf{C}^2$, there are always at least 2 holomorphic critical points. If $\rho(R)$ is positive-definite, there are at least 3 nonholomorphic critical points $\{Q_i\}$ with $\{Q_i + JQ_i\}$ linearly independent.*

PROOF. The first statement is clear. For the second, note that each nonholomorphic critical point Q satisfies
$$Q + JQ \in \{\xi \in u(V)|\langle RI, \xi \rangle = 0\} = RI^{\perp}.$$
Also, each $\xi \in RI^{\perp}$, up to a scalar factor, is of the form $Q + JQ$ for some plane $Q \in G(2, V) - (\pm\mathbf{C}P(V))$, since $\xi = av_1 \wedge v_{1^*} + bv_2 \wedge v_{2^*}$ for an appropriate choice of unitary basis. $\rho(R)$ positive-definite implies that $ab < 0$. Choose a basis so that $a > 0$, $b < 0$. Then, if $Q = (\sqrt{a}\, v_1 + \sqrt{-b}\, v_{2^*}) \wedge (\sqrt{a}\, v_{1^*} + \sqrt{-b}\, v_2)$, $Q + JQ = 2\xi$. Thus $RI^{\perp}$ is in essence the space of possible nonholomorphic critical points of R. Let $A: RI^{\perp} \to I^{\perp}$ $(= \{\xi \in u(V)|\langle \xi, I \rangle = 0\})$ be the orthogonal projection, $A(\xi) = \xi - \frac{1}{2}\langle \xi, I \rangle I$. As $\langle RI, I \rangle > 0$, A is an isomorphism.

The operator $RA^{-1}: I^{\perp} \to I^{\perp}$ is symmetric, since, for $\alpha, \beta \in RI^{\perp}$, $\langle RA^{-1}(A\alpha), A\beta \rangle = \langle R\alpha, A\beta \rangle = \langle R\alpha, \beta \rangle = \langle \alpha, R\beta \rangle = \langle A\alpha, RA^{-1}(A\beta) \rangle$. Clearly any eigenvector ξ of RA^{-1} is of the form $\xi = \lambda A(Q + JQ)$, where Q is a critical plane of r_R. As $\dim(I^{\perp}) = 3$ the corollary is finished. $\square$

3. Proof of Theorem 1.1. Let R be as in the statement of the theorem, with critical planes $\{P_1, P_2, Q_1, Q_2, Q_3\}$ as given in Corollary 2.3. If R' is another

operator with the same critical points and values, $R'I^\perp = RI^\perp$; moreover, $R - R' = T$ is zero on $RI^\perp$, as $(R - R')A^{-1} = 0$ (A is the same for both operators). Thus, $T(\xi) = \mu < \xi, RI > RI$ as T is a symmetric operator on $u(V)$ with RI as the only possible nonzero eigenvector. However, $0 = \langle TP_1, P_1 \rangle = \mu \langle P_1, RI \rangle^2$ with $\langle RI, P_1 \rangle > 0$, thus $T = 0$. $\square$

REMARK. It is possible to extend this result to all proper $R \in \mathfrak{K}(V)$, but the proof is extremely inelegant [1].

4. An application. It is not difficult to apply the general techniques of this analysis, though not necessarily the normal form itself, to establish the classical Hopf conjecture in dimension 4 for Kähler manifolds, that nonnegative curvature implies nonnegative Euler characteristic. Indeed, one need only repeat Milnor's well-known proof in the general dimension 4 case, applied to any one critical point. Alternately, it is straightforward to show that the algebraic properties of operators with positive curvature are quite restricted.

PROPOSITION 4.1. *If* $V \simeq \mathbf{C}^2$, *any proper* $R \in K(V)$ *with nonnegative sectional curvature is positive semidefinite.*

PROOF. If R is not positive semidefinite, let ξ be an eigenvector with negative eigenvalue. ξ must lie in $u(V)$, so for an appropriate choice of unitary basis, $\xi = av_1 \wedge v_{1*} + bv_2 \wedge v_{2*}$. As $r_R \geq 0$, $ab < 0$. As before, choose the basis so that $a > 0$, $b < 0$. Then $Q = (\sqrt{a}\, v_1 + \sqrt{-b}\, v_{2*}) \wedge (\sqrt{a}\, v_{1*} + \sqrt{-b}\, v_2)$ will satisfy $Q + JQ = 2\xi$ and $r_R(Q) = \langle RQ, Q \rangle = \langle R\xi, \xi \rangle < 0$, contradicting the fact that $r_R \geq 0$. $\square$

An unpublished but well-known result of B. Kostant states that, if R is any proper, positive semidefinite curvature operator, then the Gauss-Bonnet integrand is nonnegative. Thus,

COROLLARY 4.2. *If* M *is a* 4 *real-dimensional Kähler manifold with nonnegative curvature, the Euler characteristic of* M *is also nonnegtive.*

REMARK. In a forthcoming article, *Curvature and Euler characteristic for six-dimensional Kähler manifolds*, the author is able to verify this conjecture for six-dimensional Kähler manifolds as well, using techniques developed in [2].

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