

ALMOST DISJOINT REFINEMENT OF FAMILIES OF SUBSETS OF N

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ABSTRACT. Without any set-theoretic assumptions, we prove that every uniform ultrafilter on the set N of all natural numbers has a Comfort system, that is, an almost disjoint refinement. Moreover, we describe one type of ideal such that the family of all subsets of N that are not contained in it has an almost disjoint refinement.

1. The problem and the theorems. For a cardinal number $\nu > 1$, Hechler has generalized Pierce's notion of a ν -point to a ν -set of a topological space. A nonempty subset S of a topological space X is called a ν -set if there exists a family of ν pairwise disjoint open sets, each of which contains S in its closure. A point $p \in X$ is a ν -point of X if the singleton $\{p\}$ is a ν -set. We concentrate on the space $\beta N - N = N^*$ of uniform ultrafilters on the set N of all natural numbers.

The problem whether each point of N^* is a 2^ω -point or, more generally, whether each nowhere dense subset of N^* is a 2^ω -set, has a little longer history (cf. Pierce [P], Hindman [H], Comfort [CH], van Douwen [vD], Roitman [R], Kunen [K], Szymanski [Sz], Hechler [H], Frankiewicz [BF] and others). Short historical remarks can be found in [CH] or [BF].

1.1 Without any additional set-theoretic assumptions, we shall prove that every point of N^* is a 2^ω -point. This gives an affirmative answer to a problem raised by Comfort and Hindman [CH]. We also describe a type of nowhere dense subsets of N^* that are 2^ω -sets. The main problem of Hechler's paper [H], whether every nowhere dense subset of N^* is a 2^ω -set, remains open.

1.2. For a set A let $[A]^\omega$ be the set of all denumerable subsets of A ; the notation $A \subseteq^* B$ means that $A - B$ is finite. We say that a family $\{A_\alpha: \alpha < \nu\}$ of subsets of a set X is a tower on X of length ν if $A_\alpha \subseteq^* A_\beta$ for $\alpha > \beta$. We say that a set C is a selector of a family $\{q_n: n \in \omega\}$ if C is infinite, $C \subseteq \cup \{q_n: n \in \omega\}$, and $|C \cap q_n| < 1$ for $n \in \omega$. \mathcal{I}_F denotes the ideal of all finite subsets of N . In this paper all ideals are assumed to be proper and to contain \mathcal{I}_F . Sets from $\mathcal{I}^+ = \mathcal{P}(N) - \mathcal{I}$ are called large sets with respect to \mathcal{I} for an ideal \mathcal{I} .

1.3. We shall deal with families $\mathcal{C} \subseteq [N]^\omega$ and we look for \mathcal{C} that have an almost disjoint refinement (ADR) i.e. a family $\{C_X: X \in \mathcal{C}\}$ such that

- (i) $C_X \in [X]^\omega$,
- (ii) for $X \neq Y$ the set $C_X \cap C_Y$ is finite.

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Recall that $\mathcal{P} \subseteq [N]^\omega$ is a MAD family on N if \mathcal{P} is an infinite maximal family of pairwise almost disjoint infinite subsets of N .

The following facts are well known.

(i) A family $\mathcal{A} \subseteq [N]^\omega$ has an ADR iff there is a MADF \mathcal{P} such that for every $A \in \mathcal{A}$ we have

$$|\{X \in \mathcal{P} : X \cap A \text{ is infinite}\}| = 2^\omega.$$

(ii) Let \mathcal{U} be a uniform ultrafilter on N . Then \mathcal{U} as a point of N^* is a 2^ω -point of N^* iff there is an ADR for \mathcal{U} .

1.4. Following Mathias [M] we shall say that an ideal \mathcal{G} on N is tall if for all $X \in [N]^\omega$ there is a $Y \in [X]^\omega$ with $Y \in \mathcal{G}$. Let \mathcal{P} be a MAD family; then $\mathcal{G}(\mathcal{P})$ denotes the ideal generated by $\mathcal{G}_F \cup \mathcal{P}$.

DEFINITION. Let $Q = \{q_n : n \in \omega\}$ be a partition on N into infinitely many (finite or infinite) pieces such that for all $k \in \omega$ there are infinitely many q_n with at least k elements. Let $\mathcal{U}(Q)$ be the ideal generated by the union of the sets $\{X \subseteq N : (\exists k)(\forall n \in \omega)(|X \cap q_n| < k)\}$ and $\{q_n : n \in \omega\}$. In [M] it is shown that the ideals $\mathcal{G}(\mathcal{P})$ and $\mathcal{U}(Q)$ are both tall. It is easily seen that:

(a) If \mathcal{A} is a family with an ADR, then for every $X \in [N]^\omega$ there is $Y \in [X]^\omega$ such that $[Y]^\omega \cap \mathcal{A} = \emptyset$.

(b) If \mathcal{A} is a family with an ADR then there is a MAD family \mathcal{P} such that $\mathcal{A} \subseteq \mathcal{G}^+(\mathcal{P})$.

(c) Tall ideals correspond to open dense subsets of N^* that are not the whole space. If \mathcal{G} is a tall ideal then \mathcal{G}^+ has an ADR iff the complement of the open set corresponding to \mathcal{G} is a 2^ω -set.

(d) The extremal problem whether for every MADF \mathcal{P} there exists an ADR for $\mathcal{G}^+(\mathcal{P})$ is equivalent to the above-mentioned problem of Hechler.

1.5. THEOREM A. *Let Q be a partition of N as in Definition 1.4. Then the family $\mathcal{U}^+(Q)$ has an ADR.*

As a straightforward corollary we obtain that every nonselective uniform ultrafilter on N has an ADR. For ultrafilters however we shall prove a little more.

1.6. THEOREM B. *Let \mathcal{F} be a uniform ultrafilter on N and \mathcal{P} a MAD family of N with $\mathcal{P} \cap \mathcal{F} = \emptyset$. Then there is an ADR for \mathcal{F} which consists of large sets with respect to the ideal $\mathcal{G}(\mathcal{P})$.*

1.7. COROLLARY. *Since the ideal \mathcal{F}^* dual to a uniform ultrafilter \mathcal{F} is tall, there is a MADF $\mathcal{P} \subseteq \mathcal{F}^*$. Thus every uniform ultrafilter \mathcal{F} has an ADR.*

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2. Proofs of the theorems. We begin with some cardinal characteristics.

2.1. For functions from N to N consider the preordering $f <^* g$ iff $\{n : f(n) > g(n)\}$ is finite. The least cardinal of a family of functions that is unbounded under $<^*$ is denoted by λ . Obviously there is a family of functions $\{f_\alpha : \alpha \in \lambda\}$

unbounded under $<^*$ such that f_α 's are increasing and $\alpha < \beta$ implies $f_\alpha <^* f_\beta$. Note that every set of functions with cardinality less than λ has an $<^*$ -upper bound. Due to this fact, for every partition $\{X_n: n \in \omega\}$ of N whose members are infinite, there is a tower $\{A_\alpha: \alpha \in \lambda\}$ such that

- (i) $X_n - A_\alpha$ is finite for every n, α ;
- (ii) for any $X \in [N]^\omega$, if $\{n: |X_n \cap X| = \aleph_0\}$ is infinite, then $(\exists \alpha \in \lambda)(X \not\subseteq^* A_\alpha)$.

2.2. The following is defined in [BPS]. Let κ denote the least cardinal such that the Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_F$ of all subsets of ω modulo finite sets is not (κ, ∞) -distributive. The "Base matrix theorem" proved in [BPS] says: There is a system $\{\mathcal{P}_\alpha: \alpha \in \kappa\}$ of MAD families such that for $\alpha > \beta$, \mathcal{P}_α * -refines \mathcal{P}_β and for every $A \in [\omega]^\omega$ there is a $B \in \cup \{\mathcal{P}_\alpha: \alpha \in \kappa\}$ such that $B \subseteq A$.

2.3. Let d denote the minimal cardinal such that there is a MADF \mathcal{P} on N with $|\mathcal{P}| = d$.

LEMMA. $\omega_1 < \kappa < d$.

PROOF. In [BPS] the inequalities $\omega_1 < \kappa < \lambda$ are proved. The proof is finished by adding the known inequality $\lambda < d$; see [So].

Remember that under the assumption $d = 2^\omega$ Hechler's conjecture is known to be true [R], [H].

2.4. The following lemma plays a key role in our proofs.

LEMMA. There is a family $\mathfrak{B} \subseteq [\omega]^\omega$ such that the following conditions hold for any $u, v \in \mathfrak{B}$.

- (i) $u \cap v =^* \emptyset$ or $u \subseteq^* v$ or $v \subseteq^* u$;
- (ii) $|\{w \in \mathfrak{B}: u \subseteq^* w\}| < \kappa$;
- (iii) for any $X \in [\omega]^\omega$ there is a $w \in \mathfrak{B}$ such that $w \subseteq X$.

PROOF. Let $\{\mathcal{P}_\alpha: \alpha < \kappa\}$ be the base matrix mentioned in 2.2. Then $\mathfrak{B} = \cup \{\mathcal{P}_\alpha: \alpha < \kappa\}$ has the desired properties.

2.5. LEMMA. Assume \mathfrak{B} is as in Lemma 2.4, $\mathfrak{B}_0 \subseteq \mathfrak{B}$ such that $|\mathfrak{B}_0| < 2^\omega$. Then for every $C \in [\omega]^\omega$ there is a $u \in \mathfrak{B} - \mathfrak{B}_0$ such that $u \subseteq C$ and $(\forall v \in \mathfrak{B}_0)(v \cap u =^* \emptyset$ or $u \subseteq^* v)$.

PROOF. As there is a MAD family on ω of cardinality 2^ω , by (iii) of Lemma 2.4 we have $|\{v \in \mathfrak{B}: v \subseteq^* u\}| = 2^\omega$ for every $u \in \mathfrak{B}$. This fact with (i) of 2.4 finishes the proof.

2.6. PROOF OF THEOREM A. Let $Q = \{q_n: n \in \omega\}$ be a partition of N as in Definition 1.4. Put $\{A_\alpha: \alpha < 2^\omega\} = \mathcal{U}^+(Q)$. For A_α we shall now pick a set $c(\alpha)$ as follows. If $X = \{i \in \omega: |q_i \cap A_\alpha| = \aleph_0\}$ is infinite then we put $c(\alpha) = X$. Otherwise we pick $c(\alpha) \in [\omega - X]^\omega$ such that $i < j$ implies $|A_\alpha \cap q_i| < |A_\alpha \cap q_j|$ for $i, j \in c(\alpha)$. Let \mathfrak{B} be the Base family from Lemma 2.4. In the sequel \mathfrak{B} is used on ω as indexes of $\{q_i: i \in \omega\}$. By transfinite recursion through $\alpha < 2^\omega$ we shall define $F(\alpha) \in [A_\alpha]^\omega$ and $I(\alpha) \subseteq c(\alpha)$ such that

- (i) $I(\alpha) \in \mathfrak{B}$;
- (ii) $F(\alpha)$ is a selector for $R_\alpha = \{q_i \cap A_\alpha: i \in I(\alpha)\}$ i.e. $F(\alpha) \subseteq \cup R_\alpha$ and $|F(\alpha) \cap q_i \cap A_\alpha| < 1$ for any $i \in I(\alpha)$;

(iii) $F(\alpha)$ is almost disjoint with all $F(\beta)$, and $I(\alpha) \neq^* I(\beta)$ for $\beta < \alpha$.

For $\alpha < 2^\omega$ we put $D_\alpha = \cup R_\alpha$.

Step 0. There is $I(0) \in \mathfrak{B}$ such that $I(0) \subseteq c(0)$. As $F(0)$ take an (infinite) selector of R_0 .

Step $\alpha < 2^\omega$. For $\beta, \gamma < \alpha$ we have $F(\beta), I(\beta)$ such that $I(\beta) \neq^* I(\gamma)$ and $F(\beta) \cap F(\gamma)$ is finite for $\beta \neq \gamma$. Choose $I(\alpha)$ using 2.5 with respect to $\mathfrak{B}_0 = \{I(\beta): \beta < \alpha\}$ and $*$ -different from all $I(\beta)$. Put $\mathcal{V} = \{F(\beta) \cap D_\alpha: \beta < \alpha \text{ and } I(\alpha) \subseteq^* I(\beta) \text{ and } |F(\beta) \cap D_\alpha| = \aleph_0\} \cup (R_\alpha \cap [N]^\omega)$.

Members of \mathcal{V} are pairwise almost disjoint. According to the choice of $c(\alpha)$ and since $F(\beta)$'s are selectors, D_α cannot be $=^*$ to the union of any finite part of \mathcal{V} . By (ii) of Lemma 2.4 we have $|\mathcal{V}| < \kappa \leq d$. Hence there is $F(\alpha) \subseteq D_\alpha$, an infinite selector of R_α that is almost disjoint with all members of \mathcal{V} . We note that $I(\alpha) \cap I(\beta) =^* \emptyset$ implies $F(\alpha) \cap F(\beta) =^* \emptyset$. Hence $\{F(\alpha): \alpha < 2^\omega\}$ is an ADR for $\mathfrak{U}^+(Q)$. The proof of Theorem A is complete.

2.7. Our starting point for the proof of Theorem B is the notion of an ultrafilter's tower.

DEFINITION. A tower $\mathfrak{A} = \{A(\alpha): \alpha \in \nu\}$ is a tower of a uniform ultrafilter \mathfrak{F} if

- (i) ν is uncountable and regular;
- (ii) $\mathfrak{A} \subseteq \mathfrak{F}$;
- (iii) for any $X \in \mathfrak{F}$ there is $\alpha \in \nu$ such that $X \not\subseteq^* A(\alpha)$.

2.8. LEMMA. Assume \mathfrak{F} is a uniform ultrafilter on N . Then there is a tower of \mathfrak{F} (of uncountable length).

PROOF. It is clear that such a tower exists for P -ultrafilters. In the case of non- P -ultrafilters we can take a tower of the length λ from 2.1, where the partition is the one exemplifying the non- P -property.

2.9. LEMMA. Assume $\{A(\alpha): \alpha \in \nu\}$ is a tower of \mathfrak{F} and \mathfrak{P} is a MAD family such that $\mathfrak{F} \cap \mathfrak{P} = \emptyset$. Then

- (iv) $(\forall \alpha \in \nu)(\exists \beta > \alpha)(A(\alpha) - A(\beta) \in \mathfrak{P}^+(\mathfrak{P}))$.

PROOF. By induction we can choose an increasing sequence $\{\alpha_n: n \in \omega\}$ and a family $\{u_n: n \in \omega\}$ of different elements of \mathfrak{P} such that $(A(\alpha_i) - A(\alpha_{i+1})) \cap u_i$ is infinite. For $n \in \omega$ we have $A(\alpha_n) - \cup \{u_i: 0 \leq i < n - 1\} \in \mathfrak{F}$. Hence by (iii) of Definition 2.7 there are $\alpha_{n+1} > \alpha_n$ and $u_n \in \mathfrak{P} - \{u_0, \dots, u_{n-1}\}$ such that $(A(\alpha_n) - A(\alpha_{n+1})) \cap u_n$ is infinite. Put $\beta = \sup\{\alpha_n: n \in \omega\}$.

2.10. PROOF OF THEOREM B. Let \mathfrak{F} be a uniform ultrafilter and \mathfrak{P} a MADF with $\mathfrak{P} \cap \mathfrak{F} = \emptyset$. Assume $\{A(\alpha): \alpha < \nu\}$ is a tower of \mathfrak{F} with $A(\alpha) - A(\beta) \in \mathfrak{P}^+(\mathfrak{P})$ for $\alpha < \beta$. We put $\nu(\omega) = \{\alpha \in \nu: \text{cf}(\alpha) = \omega\}$. For every $\alpha \in \nu(\omega)$ we fix an increasing sequence $\{\alpha_n: n \in \omega\}$ such that $\alpha = \sup\{\alpha_n: n \in \omega\}$. We define $q(\alpha, n) = \cap \{A(\alpha_i): 0 \leq i < n\} - (A(\alpha_{n+1}) \cup A(\alpha))$. Note that $q(\alpha, n) \in \mathfrak{P}^+(\mathfrak{P})$ and $q(\alpha, n) \cap q(\alpha, m) = \emptyset$ for all $\alpha, n \neq m$. It is easy to see by Lemma 2.9 that for every $X \in \mathfrak{F}$ there is $\alpha \in \nu(\omega)$ such that the set $\{n: X \cap q(\alpha, n) \in \mathfrak{P}^+(\mathfrak{P})\}$ is infinite.

We define $Q_\alpha = \{q(\alpha, n) : n \in \omega\}$ for $\alpha \in \nu(\omega)$. If K_α, K_β are selectors of Q_α, Q_β respectively and $\alpha \neq \beta$ then $K_\alpha \cap K_\beta$ is finite. Hence for the proof of Theorem B it suffices to show that the family $\mathfrak{S}(Q_\alpha) = \{X \subseteq N : |\{n : q(\alpha, n) \cap X \in \mathcal{G}^+(\mathcal{P})\}| = \aleph_0\}$ has an ADR consisting of selectors of Q_α which are large sets. The argument is now similar to the one used in 2.6. Let $Q = \{q_n : n \in \omega\} = Q_\alpha$ for $\alpha \in \nu(\omega)$ and let $\{D(\alpha) : \alpha < 2^\omega\}$ be a numbering of $\mathfrak{S}(Q)$. We put $c(\alpha) = \{i : D(\alpha) \cap q_i \in \mathcal{G}^+(\mathcal{P})\}$. By transfinite recursion we define sets $I(\alpha), F(\alpha)$ such that

- (i) $I(\alpha) \in \mathfrak{B}$, where \mathfrak{B} is the Base family from Lemma 2.4 and $I(\alpha) \subseteq c(\alpha)$;
- (ii) $F(\alpha)$ is a selector for $\{q_i \cap D_\alpha : i \in I(\alpha)\}$ and $F(\alpha) \in \mathcal{G}^+(\mathcal{P})$;
- (iii) for $\beta < \alpha$, $F(\alpha) \cap F(\beta)$ is finite and $I(\beta) \cap I(\alpha) =^* \emptyset$ or $(I(\alpha) \subseteq^* I(\beta)$ and $I(\alpha) \neq^* I(\beta))$.

In the step $\alpha < 2^\omega$ we choose $I(\alpha) \in \mathfrak{B}$ using Lemma 2.5. The $F(\beta)$'s are selectors and hence they determine partial functions f_β on ω . We set $\mathcal{V} = \{f_\beta \cap (I(\alpha) \times D(\alpha)) : \beta < \alpha, \text{ and } I(\alpha) \subseteq^* I(\beta)\}$. As $|\mathcal{V}| < \kappa < \lambda$ there is a function $f : I(\alpha) \rightarrow N$ that is an $<^*$ -upper bound for \mathcal{V} . We note that for any infinite family of pairwise disjoint large sets there is a large selector. Hence we may take $F(\alpha)$ as a large selector of the family $\{D_\alpha \cap q_i - \{n : n < f(i)\} : i \in I(\alpha)\}$. It is obvious that $F(\alpha) \cap F(\beta)$ is finite for $\beta < \alpha$.

This completes the proof.

3. Remarks and problems.

3.1. We do not know if the following observation is known. Let us consider a MAD family on the set Q of all rational numbers in the unit interval $[0, 1]$ of the real line. Then there is a MAD family \mathcal{P} on Q such that for any set $A \subseteq Q$ that has infinitely many accumulation points in the space $[0, 1]$ there exists $B \in \mathcal{P}$ with $B \subseteq A$. This fact follows from Theorem A.

3.2. Let $s = \{a_n : n \in N\}$ be a sequence of positive reals with $\lim a_n = 0$ and $\sum a_n = \infty$. Let us consider the ideal $\mathcal{Y}(s) = \{X \subseteq N : \sum \{a_n : n \in X\} < \infty\}$. This type of ideal seems to be similar to the ideal of type $\mathcal{Y}(Q)$ from Definition 1.4, where Q is partition of N consisting of finite sets. But we do not know whether $\mathcal{Y}^+(s)$ has an ADR.

3.3. Does the assumption "every $<^*$ -cofinal subset of functions from N to N has cardinality 2^ω " imply Hechler's conjecture?

3.4. Consider the Boolean algebra $B = \mathcal{P}(N)/\mathcal{I}_F$. Corollary 1.7 is equivalent to the statement "every filter base on B of cardinality at most 2^ω has a disjoint refinement". This statement cannot be strenghtened to the completion \bar{B} of the algebra B . Using a result of Kunen, van Mill and Mills [KvMM] in [BSV] have proved the following.

If $2^\tau < 2^\omega$ for all $\tau < 2^\omega$ then there is an ultrafilter on \bar{B} with a base of cardinality 2^ω . Then there is no disjoint refinement on \bar{B} for any base of this ultrafilter.

3.5. K. Kunen, using an observation from [BF], proved the following generalization of a result in [BF]. If X is any compact space in which nonempty G_δ sets have nonempty interior, then very nonisolated point in X is an ω_1 -point. He also has remarked that for the above class of spaces we cannot replace ω_1 by 2^ω .

Let us consider only spaces that moreover have no isolated point. Are there any simple conditions on such spaces that imply "every point is a 2^ω -point"? We note that for arbitrary $\tau > \omega$, every point of the space $\beta(\tau) - \tau$ is a 2^ω -point.

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