ON UNIONS OF \( \nu \)-EMBEDDED SETS

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ABSTRACT. Let \( A \) be a realcompact and \( C \)-embedded subspace of a space \( X \) and let \( B \) be a \( \nu \)-embedded subspace of a space \( X \). Then \( A \cup B \) is \( \nu \)-embedded in \( X \).

1. Introduction. All spaces considered in this paper are assumed to be completely regular Hausdorff. For a space \( X \), \( \nu X \) denotes the Hewitt realcompactification of \( X \). A subset \( S \) of \( X \) is said to be \( C \)- (resp. \( C^* \)-) embedded in \( X \) if every real-valued continuous (resp. bounded continuous) function on \( S \) can be extended to a real-valued continuous function over \( X \), and \( S \) is \( z \)-embedded in \( X \) if for each zero-set \( Z \) of \( S \), there exists a zero-set \( Z' \) of \( X \) such that \( Z = Z' \cap S \). Clearly, every \( C \)-embedded subset is \( C^* \)-embedded, and every \( C^* \)-embedded subset is \( z \)-embedded. In [1], R. L. Blair introduced the concept of \( \nu \)-embedding as a generalization of \( z \)-embedding as follows. \( S \) is \( \nu \)-embedded in \( X \) if the extension \( \tau : \nu S \to \nu X \) of the inclusion map \( i : S \to X \) is a homeomorphism of \( \nu S \) into \( \nu X \). Clearly, every realcompact subspace of \( X \) is \( \nu \)-embedded in \( X \). We denote that \( S \) is \( \nu \)-embedded in \( X \) by \( \nu S \subset \nu X \). In [1], R. L. Blair proved the following results.

(A) Assume that \( X \) is locally compact and that \( S = (\bigcup_{i=1}^{n} A_i) \cup B \), where \( B \) is \( G_\delta \)-closed and \( \nu \)-embedded in \( X \) (so that \( \nu B \subset \nu X \)) and each \( A_i \) is realcompact and \( C \)-embedded in \( X \). Then \( \nu S = (\bigcup_{i=1}^{n} A_i) \cup \nu B \) (so \( S \) is \( \nu \)-embedded in \( X \)).

(B) In any locally compact space \( X \), the union of a compact set with a cozero-set is \( \nu \)-embedded in \( X \).

R. L. Blair asked whether in both cases above, the hypothesis of local compactness can be omitted. The purpose of this paper is to answer this question affirmatively. Furthermore, we shall show that the hypothesis of \( G_\delta \)-closedness of \( B \) can be omitted.

Hereafter, \( C(X) \) (resp. \( C^*(X) \)) denotes the set of all real-valued continuous (resp. bounded continuous) functions on a space \( X \), \( N \) the space of natural numbers and \( R \) the space of real numbers. For realcompact spaces, the reader is referred to [1] and [2].

2. On unions of \( \nu \)-embedded sets. Firstly we shall show the following lemma which is needed for our study.

**Lemma 2.1.** Let \( X \) be a space and \( A, B \) subspaces of \( X \). If \( A \) is closed in \( \nu X \) and \( B \) is \( \nu \)-embedded in \( X \), then \( A \cup B \) is \( C \)-embedded in \( A \cup \nu B \).
Proof. Let us denote $A \cup B$ by $S$. Let $f \in C(S)$. Then there exists a $g \in C(vB)$ which is the extension of $f|B$. Let us define a real-valued function $h: A \cup vB \to R$ as follows.

$$h(p) = \begin{cases} f(p) & \text{if } p \in A, \\ g(p) & \text{if } p \in vB. \end{cases}$$

Since $(A \cap vB) \cup B$ is $C$-embedded in $vB$, and $B$ is dense in $(A \cap vB) \cup B$, $f|((A \cap vB) \cup B) = g|(A \cap vB)$. Hence the function $h$ is well defined. In order to prove this lemma, it suffices to show that $h$ is a continuous function on $A \cup vB$. If $p \in (A \cup vB) - A$, it is easy to prove that $h$ is continuous at $p$. So we consider $p \in A$. Let $G$ be an open neighborhood of $h(p)$ in $R$. Then there exists an open neighborhood $H$ of $h(p)$ in $R$ such that $\text{cl}_R H \subset G$. Since $p \in A$, $h(p) = f(p)$. By the continuity of $f$, there exists an open neighborhood $U$ of $p$ in $S$ such that $f(U) \subset H$. Then there exists an open set $U^*$ in $A \cup vB$ such that $U^* \cap S = U$. Let $U^*_1 = U^* \cap A$, $U^*_2 = U^* \cap vB$ and $U_2 = U^* \cap B$. Then $U^*_1 = U^*_1 \cup U^*_2$ and $U^*_2 \subset \text{cl}_{vB} U_2$. So $h(U^*_1) = f(U^*_1) \subset G$ and

$$h(U^*_2) = g(U^*_2) \subset g(\text{cl}_{vB} U_2) \subset cl_R g(U_2) = cl_R f(U_2) \subset cl_R H \subset G.$$ 

Hence $h(U^*) = h(U^*_1) \cup h(U^*_2) \subset G$. So $h$ is continuous at $p$. Hence $h$ is a continuous function on $A \cup vB$, and the proof of Lemma 2.1 is completed.

Corollary 2.2. Let $X$ be a space and $A, B$ subspaces of $X$. If $B$ is $v$-embedded in $X$ and $A$ is realcompact and $C$-embedded in $X$, then $A \cup B$ is $C$-embedded in $A \cup vB$.

Proof. Since every realcompact and $C$-embedded subset of $X$ is closed in $vX$ [2, 8.10(a)], this follows from Lemma 2.1.

Theorem 2.3. Let $X$ be a space and $A, B$ subspaces of $X$. If $B$ is $v$-embedded in $X$ and $A$ is realcompact and $C$-embedded in $X$, then $v(A \cup B) = A \cup vB$. So $A \cup B$ is $v$-embedded in $X$.

Proof. By [1, Lemma 6.3], $A \cup vB$ is realcompact, and by Corollary 2.2, $A \cup B$ is $C$-embedded in $A \cup vB$. Since $A \cup B$ is dense in $A \cup vB$, $v(A \cup B) = A \cup vB$.

Corollary 2.4. Let $X$ be a space and $A_1, \ldots, A_n$, $B$ subspaces of $X$. If $B$ is $v$-embedded in $X$ and each $A_i$ is realcompact and $C$-embedded in $X$, then $v((\bigcup_{i=1}^n A_i) \cup B) = (\bigcup_{i=1}^n A_i) \cup vB$. So $(\bigcup_{i=1}^n A_i) \cup B$ is $v$-embedded in $X$.

Corollary 2.5. In any space $X$, the union of a compact subset with a $v$-embedded subset is $v$-embedded in $X$.

Since every cozero-set of $X$ is $v$-embedded in $X$ [1, Theorem 5.1], Corollary 2.4 and Corollary 2.5 are the answer to R. L. Blair's question quoted in the introduction. Next, we shall show that in Theorem 2.3 "$C$-embedded" cannot be weakened to "$C^*$-embedded". The following example is a modification of the Dieudonné plank in [3].

Example 2.6. There exists a non-realcompact space which is the union of a realcompact subspace with a $C^*$-embedded Lindelöf subspace. Hence we cannot
replace “C-embedded” by “C*-embedded” in Theorem 2.3.

For any ordinal $\alpha$, $W(\alpha)$ denotes the space of all ordinals less than $\alpha$ with the usual order topology. Let $\omega_1$ be the first uncountable ordinal. Let $D$ be the subset of $W(\omega_1)$ consisting of all isolated ordinals, and $\tilde{D} = D \cup \{\omega_1\}$. Then $D$ is a discrete space of a nonmeasurable cardinal (so $D$ is realcompact) and $\tilde{D}$ is the one-point Lindelöfication of $D$. Let $\beta N$ be the Stone-Čech compactification of $N$. We construct $X$ as follows.

$$X = \tilde{D} \times \beta N - (\{\omega_1\} \times (\beta N - N)).$$

We consider $X$ as a subspace of the product space $\tilde{D} \times \beta N$. Let $A = D \times \beta N$ and $B = (\omega_1) \times N$. Then $X = A \cup B$ and $A$ is realcompact and $B$ is Lindelöf. The space $X$ satisfies the following assertions.

**Assertion 1.** $\nu X = \tilde{D} \times \beta N$, so $X$ is not realcompact.

**Proof.** Let $f \in C(X)$. Since each point of $B$ is a $P$-point in $X$, we can choose a neighborhood $U$ of $\omega_1$ in $\tilde{D}$ such that $f$ is constant on $U \times \{n\}$ for each $n \in N$. Since $U \times N$ is dense in $U \times \beta N - (\omega_1) \times (\beta N - N)$, $f$ is constant on $(U - (\omega_1)) \times \{p\}$ for each $p \in \beta N - N$. Let $p_U$ be the constant value of $f$ on $(U - (\omega_1)) \times \{p\}$ for each $p \in \beta N - N$. We define a real-valued function $\tilde{f}: \tilde{D} \times \beta N \rightarrow R$ as follows.

$$\tilde{f}|X = f \quad \text{and} \quad \tilde{f}(\omega_1, p) = p_U \quad \text{for each } p \in \beta N - N.$$

Then $\tilde{f}$ is a continuous extension of $f$ over $\tilde{D} \times \beta N$. Hence $X$ is $C$-embedded in $\tilde{D} \times \beta N$. Since $\tilde{D} \times \beta N$ is realcompact and $X$ is dense in $\tilde{D} \times \beta N$, $\nu X = \tilde{D} \times \beta N$.

**Assertion 2.** $B$ is $C^*$-embedded in $X$.

**Proof.** Let $f \in C^*(B)$. Then there exists a $g \in C^*(\{\omega_1\} \times \beta N)$ such that $g|B = f$. Let us define a real-valued function $h: \tilde{D} \times \beta N \rightarrow R$ as follows.

$$h(\alpha, p) = g(\omega_1, p) \quad \text{for all } \alpha \in \tilde{D}, p \in \beta N.$$ 

Then $h|X$ is a continuous extension of $f$ over $X$.

**References**


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