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ON THE HEIGHT OF THE FIRST STIEFEL-WHITNEY CLASS

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ABSTRACT. Let $G_k(\mathbf{R}^{n+k})$ denote the grassmann manifold of k-planes in real $(n+k)$-space and $w_1 \in H^1(G_k(\mathbf{R}^{n+k}); \mathbf{Z}_2)$ the first Stiefel-Whitney class of the universal bundle. We determine, for many (k, n), the exact height of w_1 in the cohomology ring. We also indicate the combinatorial significance of the complex analogue of these computations.

Let $G_k(\mathbf{R}^{n+k})$ denote the grassmann manifold of k-planes in real $(n+k)$-space. We have previously determined estimates on the height of the first Stiefel-Whitney class w_1 of the universal k-plane bundle over $G_k(\mathbf{R}^{n+k})$ [2]. Indeed, we showed that if $1 < k \leqslant n$ and $2^s < n + k \leqslant 2^{s+1}$, then the height of w_1 is $2^{s+1} - \alpha$, where α is either 1 or 2 (unless $k = 2^t = n$, in which case we can only assert $1 \leqslant \alpha \leqslant 2^t$). As the codimension n increases, it is clear that the height of w_1 is nondecreasing; hence it suffices to find where in the intervals of n: $[2^s + 1 - k, 2^{s+1} - k]$ the value of α jumps down. For $k = 2$, it was shown that α is identically 2 [2]. In this note, we settle the problem for $k = 3, 4, 5$ and for $k \geqslant 6$ (except for finitely many n).

We use the Schubert calculus description of the mod 2 cohomology of $G_k(\mathbf{R}^{n+k})$ [1]. Additively, a $\mathbf{Z}_2$-basis is provided by the Schubert symbols $(a_1, \ldots, a_k)$ where $0 \leqslant a_1 \leqslant \ldots \leqslant a_k \leqslant n$. Indeed, the tangential and normal Stiefel-Whitney classes of the universal bundle are themselves represented by Schubert symbols; namely

$$w_i = \Big(0, \ldots, 0, \underbrace{1, \ldots, 1}_{i}\Big), \qquad \bar{w}_j = (0, \ldots, 0, j),$$

where $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$. One often refers to the $\bar{w}_j$ as special Schubert symbols. They are special, from one point of view, since there is a combinatorial formula describing their multiplication by an arbitrary Schubert symbol; namely

$$\bar{w}_j(a_1, \ldots, a_k) = \sum (b_1, \ldots, b_k),$$

where the sum ranges over all k-tuples $(b_1, \ldots, b_k)$ such that $a_i \leqslant b_i \leqslant a_{i+1}$, $1 \leqslant i \leqslant k$ (where $a_{k+1} = n$) and $b_1 + \cdots + b_k = j + a_1 + \cdots + a_k$. This is called the Pieri intersection formula. We now begin with

LEMMA 1. *In* $H^*(G_3(\mathbf{R}^{n+3}); \mathbf{Z}_2)$,

$$w_1^j = \sum_{p,q} \left[\binom{p+q-1}{q} \binom{j}{p+q-2} + \binom{p+q+1}{q+1} \binom{j+1}{p+q} \right] (p, q, j-p-q).$$

PROOF. Use the Pieri formula and induction; see [3].

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LEMMA 2.

$$\binom{2^s+j}{k} \equiv 0 \pmod 2 \quad \text{if } j+1 \leq k \leq 2^s - 1.$$

PROOF. The result is immediate for $j = 0$; we then invoke induction and

$$\binom{2^s+j}{k} = \binom{2^s+j-1}{k} + \binom{2^s+j-1}{k-1} \equiv 0 \pmod 2$$

for $k \geq j+1$. This completes the proof.

It is now easy to show

PROPOSITION 3. *In* $H^*(G_3(\mathbf{R}^{2^s+2}); \mathbf{Z}_2)$,

$$w_1^{2^{s+1}-1} = \sum_{i=1}^{2^{s-1}} (i, 2^s - i, 2^s - 1) \neq 0.$$

PROOF. By Lemma 1, this is equivalent to

$$\binom{p+q-1}{q} \equiv 0 \pmod 2$$

unless $p + q = 2^s$ (for which clearly the binomial coefficient is odd). By the Schubert condition, $q \leq 2^{s+1} - 1 - p - q$, i.e. $q \leq 2^s - (p + 1/2)$. Also $p + q \geq 2^s + 1$, or $q \geq 2^s - p + 1$; so we can write $q = 2^s - (p-1) + j$, $0 \leq j \leq (p - 1/2)$. Reindexing by j, we see

$$\binom{p+q-1}{q} = \binom{2^s+j}{2^s-(p-1)+j}.$$

By Lemma 2, it suffices to check that $j + 1 \leq 2^s - (p-1) + j \leq 2^s - 1$. The first inequality becomes $p \leq 2^s$ which is immediate from the Schubert condition. The second inequality asserts $j + 2 \leq p$. So when j is maximized this becomes $p \geq 3$. This is also immediate, since if $p < 3$, it is easy to check that $p + q = 2^s$. This completes the proof.

In [2, Lemma 4.6] we showed $w_1^{2^{s+1}-1} = 0$ in $H^*(G_3(\mathbf{R}^{2^s+1}); \mathbf{Z}_2)$, so that for $k = 3$, $\alpha = 2$ at the first place in the interval and by Proposition 3, $\alpha = 1$ thereafter. To settle the case $k = 4$ we have

PROPOSITION 4. *For* $s \geq 3$, *in* $H^*(G_4(\mathbf{R}^{2^s+1}); \mathbf{Z}_2)$,

$$w_1^{2^{s+1}-1} \neq 0.$$

PROOF. By Propositions 3.1 and 3.2 of [2], $w_1^{2^s} = (1, 1, 1, 2^s - 3)$. So it suffices to compute the height of w_1 inside of $H^*(G_3(\mathbf{R}^{2^s-1}); \mathbf{Z}_2)$. Now we have the inclusion

$$i\colon G_3(\mathbf{R}^{2^s-1+2}) \subset G_3(\mathbf{R}^{2^s-1}).$$

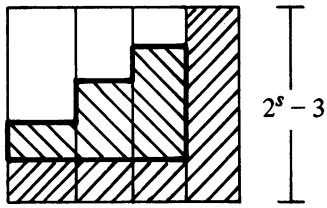
In the former space, by Proposition 3, $w_1^{2^s-1} \neq 0$; hence also in the latter space. It is easy to see there is an injection

$$e\colon H^*(G_3(\mathbf{R}^{2^s-1}); \mathbf{Z}_2) \to H^*(G_4(\mathbf{R}^{2^s+1}); \mathbf{Z}_2)$$

defined on the Schubert basis by

$$e(a_1, a_2, a_2) = (a_1 + 1, a_2 + 1, a_3 + 1, 2^s - 3).$$

Graphically, we embed the 3-cycle in the upper left-hand corner



where the heights of the column indicate the entries of the Schubert cycle. The Pieri formula then implies

$$w_1^{2^s+x} = e(w_1^x)$$

where the locations of the w_1's are determined by context. Since e is injective and $w_1^{2^s-1} \neq 0$, we conclude

$$w_1^{2^{s+1}-1} = w_1^{2^s+2^s-1} = e(w_1^{2^s-1}) \neq 0.$$

Proposition 4 immediately yields that α is identically 1 for the case $k = 4$.

PROPOSITION 5. *For $s \geqslant 3$, in $H^*(G_5(\mathbf{R}^{2^s+1}); \mathbf{Z}_2)$,*

$$w_1^{2^{s+1}-1} \neq 0.$$

PROOF. The argument is entirely analogous to that of Proposition 4, except we use the inclusion

$$G_4(\mathbf{R}^{2^{s-1}+1}) \subset G_4(\mathbf{R}^{2^s-1}).$$

Hence, we must assume $s \geqslant 4$. For the case $s = 3$, we observe $G_5(\mathbf{R}^9) \cong G_4(\mathbf{R}^9)$, so the result again follows from Proposition 4.

Finally, we come to

PROPOSITION 6. *For $s \geqslant k - 2$, in $H^*(G_k(\mathbf{R}^{2^s+1}); \mathbf{Z}_2)$,*

$$w_1^{2^{s+1}-1} \neq 0.$$

PROOF. Use the inclusion $G_{k-1}(\mathbf{R}^{2^{s-1}+1}) \subset G_{k-1}(\mathbf{R}^{2^s-1})$ and induction.

We summarize our results in the following table:

$k \backslash n$	$2^s + 1 - k$	$2^s + 2 - k$	$\cdots$	$2^{s+1} - 1 - k$	$2^{s+1} - k$
$k = 2$	2	2	$\cdots$	2	2
$k = 3$	2	1	$\cdots$	1	1
$k = 4$	1	1	$\cdots$	1	1
$k = 5$	1	1	$\cdots$	1	1
$k \geqslant 6$	1	1	$\cdots$	1	1

$(s \geqslant k - 2)$

Values of $\alpha(k, n)$, where $k < n$, i.e. $k < 2^{s-1}$; height of $w_1 = 2^{s+1} - \alpha(k, n)$ in the mod 2 cohomology of $G_k(\mathbf{R}^{n+k})$.

REMARK. One can ask similar questions for the first Chern class c_1 in $H^2(G_k(\mathbf{C}^{n+k}); \mathbf{Z})$. Of course, the integral version of the Pieri formula immediately implies that $c_1^{nk} \neq 0$. (One could also argue that c_1 is a Kähler 2-form, so raising it to the complex dimension of the manifold is a volume element.) Now the $2nk$-cohomology group of $\mathbf{C}^{n+k}$ is generated by the fundamental class $(n, n, \ldots, n)$; so $c_1^{nk} = N(n, \ldots, n)$ for some integer N. It is an immediate consequence of the "hook formula" that

$$N = \frac{(nk)!}{1^1 2^2 \cdot \ldots \cdot k^k (k+1)^k \cdot \ldots \cdot n^k (n+1)^{k-1} \cdot \ldots \cdot (n+k-1)^1}.$$

The number on the right can also be interpreted as the degree of a certain irreducible representation of the symmetric group on nk letters. It might be interesting to try a similar computation for G/P, where G is a complex Lie group and P is a maximal parabolic.

Finally, note that if $k = 2$, N is the nth Catalan number $(n+1)^{-1}\binom{2n}{n}$.

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