A COMPARISON OF CHEHATA’S AND CLIFFORD’S ORDINALLY SIMPLE ORDERED GROUPS

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Abstract. Chehata and Clifford gave the two most well-known examples of ordinally simple ordered groups. Chehata’s example is also algebraically simple. These two examples are shown to be similar. Clifford’s ordered group is shown to have a nonabelian algebraically simple subgroup.

An ordinally simple ordered group (o-group) is one that has no proper normal convex subgroups. Any subgroup of the real numbers is ordinally simple. Nonabelian examples of ordinally simple ordered groups are more difficult to produce. B. H. Neumann [6] constructed the first and Clifford [3] later gave a very straightforward example. Chehata [1] gave an example which is even simple as a group. On the surface, Clifford’s and Chehata’s groups seem very different. Recently, Chehata [2] has even elaborated on the differences. In this paper, we take the opposite view and emphasize the similarities. In fact each of the two groups is a rather large subgroup of a certain ordinally simple ordered group which is defined in a natural way. This representation of Clifford’s group makes it easy to see that Clifford’s group contains an algebraically simple subgroup.

Clifford’s o-group, CL, is defined as the group generated by \( \{g_r\} \), \( r \in \mathbb{Q} \), such that \( g_s^{-1}g_sg_r = g_{s+r}/2 \) if \( s < r \). Each \( x \in CL \) then has a unique normal form \( x = g_{r_1}g_{r_2} \cdots g_{r_k} \) where \( r_1 < r_2 < \cdots < r_k \) and \( n_i \neq 0 \ \forall i \). We order this group by calling an element positive if \( n_k > 0 \).

Two positive elements, \( x \) and \( y \), of an o-group are said to be archimedeanly equivalent (written \( x \sim y \)) if \( \exists m, n \in \mathbb{Z} \) such that \( x^m > y \) and \( y^m > x \). Notice that for two positive elements of CL, \( x = g_{r_1}g_{r_2} \cdots g_{r_k} \) and \( y = g_{s_1}g_{s_2} \cdots g_{s_k} \), \( x \sim y \) iff \( r_k = s_j \). This group is clearly ordinally simple but is not simple since the set of elements whose sum of exponents is divisible by a fixed \( n \) is a normal subgroup, as one can easily verify.

Chehata’s o-group, CH, is defined as the group of order preserving permutations of the rationals, \( \mathbb{Q} \), that consist of a finite number of linear pieces and have bounded support. (\( \alpha \in \mathbb{Q} \) is in the support of a permutation \( f \) if \( \alpha f \neq \alpha \).) We order CH by calling an element positive if its rightmost nonidentity linear piece has slope less than 1. (Equivalently, if for \( \alpha \) in the domain of the rightmost nonidentity linear piece of \( f \) we have \( \alpha f > \alpha \), then \( f \) is positive.) The positive cone given here is different from the one originally given in [1], but is equivalent. Also, Chehata’s...
group can be so defined for any ordered field in place of \( \mathbb{Q} \). Chehata showed in \([1]\) that \( CH \) is simple.

Let \( D \) be the group of order preserving permutations of \( \mathbb{Q} \) whose supports are right bounded (that is, \( \sqrt{f} \in D \exists \alpha \in \mathbb{Q} \) such that \( \forall \beta > \alpha, \beta f = \beta \)) and consist of a finite number of linear pieces. We order \( D \) the same way as \( CH \). Notice that \( CH \) is an \( o \)-subgroup of \( D \). Also note that \( D \) is ordinally simple. For a description of more general \( o \)-groups of this type, see Dlab \([4]\).

We now show that Clifford’s \( o \)-group like Chehata’s can be realized as an \( o \)-subgroup of \( D \).

**Theorem.** \( CL \) can be \( o \)-embedded into \( D \).

**Proof.** Let \( g_r, r \in \mathbb{Q} \), be a generator of \( CL \). Define \( g_r \in D \) by

\[
(\alpha)(g_r) = \begin{cases} 
\alpha & \text{if } \alpha > r, \\
\frac{1}{2}(\alpha + r) & \text{if } \alpha < r.
\end{cases}
\]

Extend \( \varphi \) to include \( \{ g_r^{-1}, r \in \mathbb{Q} \} \), so that \( (g_r^{-1}) = (g_r \varphi)^{-1} \). That is,

\[
(\alpha)(g_r^{-1}) = \begin{cases} 
\alpha & \text{if } \alpha > r, \\
2\alpha - r & \text{if } \alpha < r.
\end{cases}
\]

A routine calculation shows that for \( s < r \):

\[
s_{(s+r)/2} = (g_r \varphi)^{-1}(g_s \varphi)(g_r \varphi). \tag{*}
\]

So for an arbitrary element \( g \in CL \) with normal form \( g = g_{r_1}^n g_{r_2}^{n_2} \ldots g_{r_k}^{n_k} \), define \( g \varphi = (g_r \varphi)^{n_r}(g_r \varphi)^{n_2} \ldots (g_r \varphi)^{n_k} \).

\( (*) \) shows that this is a well-defined homomorphism.

\( \varphi \) is order-preserving for if \( g = g_{r_1}^n g_{r_2}^{n_2} \ldots g_{r_k}^{n_k} \) is in normal form, then for \( r_{k-1} < \alpha < r_k \),

\[
(\alpha)(g \varphi) = \begin{cases} 
((2^k - 1)r_k + \alpha)/2^k & \text{if } n_k > 0, \\
2|n_k| \alpha - (2|n_k| - 1)r_k & \text{if } n_k < 0.
\end{cases}
\]

In either case this shows that the slope of the rightmost nonidentity piece is \((\frac{1}{2})^k\).

So if \( n_k > 0 \) (i.e. \( g > 0 \)), \( g \varphi \) is a positive element of \( D \). As a consequence \( \varphi \) is \( 1 \)-1.

One should note that \( (CL)\varphi \) consists of those elements of \( D \) whose slopes of the linear pieces are integral powers of \( \frac{1}{2} \). In fact, if \( g = g_{r_1}^{n_1} g_{r_2}^{n_2} \ldots g_{r_k}^{n_k} \) the linear pieces break at \( r_1, r_2, \ldots, r_k \) and the slope of the piece with domain \([r_{j-1}, r_j]\) is \((\frac{1}{2})^n\) where \( n = \Sigma_{j=1}^{n} n_j \).

From now on we freely interchange \( CL \) and \( (CL)\varphi \). Context will indicate which form of \( CL \) we use.

Let \( B \subseteq CL \) be those elements of \( CL \) with bounded support. By the above remarks, it is clear that if \( g \in B \) with \( g = g_{r_1}^{n_1} g_{r_2}^{n_2} \ldots g_{r_k}^{n_k} \) then \( \Sigma_{j=1}^{n} n_j = 0 \). Chehata \([2]\) has shown that the commutator subgroup of \( CL \) is precisely those elements whose sum of powers is 0. Hence \( B \subseteq [CL, CL] \) and in fact it is a normal subgroup. Chehata \([2]\) incorrectly states that the set of elements of \([CL, CL]\) which have the sum of their positive powers divisible by a fixed \( n \) is a normal subgroup of \([CL, CL]\). But this is not a subgroup, for example if \( n = 3 \), \( g_1^{-3} g_2^3 \) and \( g_2^{-1} g_3^{-2} g_4^3 \) are elements of \([CL, CL]\) but their product is \( g_1^{-3} g_2^3 g_3^{-2} g_4^3 \). The group \( B \) is the only
normal subgroup of \([\text{CL}, \text{CL}]\) known to the author.

**Lemma (Higman [5]).** If \((G, \Omega)\) is any permutation group that satisfies:

Whenever \(f, g, h \in G\) with \(h \neq e\) then \(\exists k \in G\) such that
\((\text{supp}(f) \cup \text{supp}(g))kh \cap (\text{supp}(f) \cup \text{supp}(g))k = \emptyset\), then \([G, G]\)
is simple.

**Theorem.** \(\text{CL}\) contains a nonabelian simple \(o\)-subgroup. In fact, the commutator subgroup of \(B\) is simple.

**Proof.** Clearly \(B\) (and \(\text{CL}\)) is 0-2 transitive (that is, if \(\alpha_1 < \beta_1\), and \(\alpha_2 < \beta_2\) then
\(\exists g \in B\) such that \(\alpha_1 g = \alpha_2\) and \(\beta_1 g = \beta_2\)). Let \(f, g, h \in B\) and suppose
\((\text{supp}(f) \cup \text{supp}(g)) \subseteq [\alpha, \beta]\). Pick \(\gamma \in \text{supp}(h)\). Either \(\gamma h > \gamma\) or \(\gamma h < \gamma\). Without loss of
generality assume the former. By 0-2 transitivity \(\exists k \in B\) such that
\(ak = \gamma\) and \(\beta k = ah\). Then \(akh = \gamma h = \beta k > (\text{supp}(f) \cup \text{supp}(g))k\) and so Higman's Lemma
says that \([B, B]\) is simple.

**Question.** Is \([B, B] = B\)? If it were, then of course the above theorem would be more appealing.

We can define other "Clifford-like" groups by adjusting the conjugation as follows. For \(0 < t < 1\), define \(\text{CL}_t\) to be the \(o\)-group generated by \(\{g_r\}, r \in \mathbb{Q}\),
such that \(g_r^{-1} g s g_r = g_{(1-t)r+s}\) for \(r > s\). Order \(\text{CL}_t\) in the same manner as \(\text{CL}\). Note
that if \(t = \frac{1}{2}\), we have Clifford's \(o\)-group. We can now embed \(\text{CL}_t\) into \(D\) in a
completely analogous fashion. The generators of \(\text{CL}_t\) will then have \(t\) as the slope
of their first nonidentity piece. The permutations determined by elements of \(\text{CL}_t\)
will then be those of \(D\) that have as slopes of their linear pieces powers of \(t\). The
theorem concerning the simple subgroup of \(\text{CL}_t\) is of course the same as for \(\text{CL}\).

**References**


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