PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, August 1980

A COMPARISON OF CHEHATA'S AND CLIFFORD'S ORDINALLY SIMPLE ORDERED GROUPS[1]

TODD FEIL

ABSTRACT. Chehata and Clifford gave the two most well-known examples of ordinally simple ordered groups. Chehata's example is also algebraically simple. These two examples are shown to be similar. Clifford's ordered group is shown to have a nonabelian algebraically simple subgroup.

An ordinally simple ordered group (o-group) is one that has no proper normal convex subgroups. Any subgroup of the real numbers is ordinally simple. Nonabelian examples of ordinally simple ordered groups are more difficult to produce. B. H. Neumann [6] constructed the first and Clifford [3] later gave a very straightforward example. Chehata [1] gave an example which is even simple as a group. On the surface, Clifford's and Chehata's groups seem very different. Recently, Chehata [2] has even elaborated on the differences. In this paper, we take the opposite view and emphasize the similarities. In fact each of the two groups is a rather large subgroup of a certain ordinally simple ordered group which is defined in a natural way. This representation of Clifford's group makes it easy to see that Clifford's group contains an algebraically simple subgroup.

Clifford's o-group, CL, is defined as the group generated by $[g_r]$, $r \in \mathbf{Q}$, such that $g_r^{-1}g_sg_r = g_{(s+r)/2}$ if $s < r$. Each $x \in \mathrm{CL}$ then has a unique normal form $x = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$ where $r_1 < r_2 < \cdots < r_k$ and $n_i \neq 0\ \forall i$. We order this group by calling an element positive if $n_k > 0$.

Two positive elements, x and y, of an o-group are said to be archimedeanly equivalent (written $x \sim y$) if $\exists m, n \in Z$ such that $x^n > y$ and $y^m > x$. Notice that for two positive elements of CL, $x = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$ and $y = g_{s_1}^{m_1}g_{s_2}^{m_2} \cdots g_{s_j}^{m_j}$, $x \sim y$ iff $r_k = s_j$. This group is clearly ordinally simple but is not simple since the set of elements whose sum of exponents is divisible by a fixed n is a normal subgroup, as one can easily verify.

Chehata's o-group, CH, is defined as the group of order preserving permutations of the rationals, $\mathbf{Q}$, that consist of a finite number of linear pieces and have bounded support. ($\alpha \in \mathbf{Q}$ is in the support of a permutation f if $\alpha f \neq \alpha$.) We order CH by calling an element positive if its rightmost nonidentity linear piece has slope less than 1. (Equivalently, if for α in the domain of the rightmost nonidentity linear piece of f we have $\alpha f > \alpha$, then f is positive.) The positive cone given here is different from the one originally given in [1], but is equivalent. Also, Chehata's

Received by the editors August 13, 1979.
AMS (MOS) subject classifications (1970). Primary 06F15.
[1]The material in this paper is from the author's Ph. D. thesis, Bowling Green State University.

group can be so defined for any ordered field in place of **Q**. Chehata showed in [1] that CH is simple.

Let D be the group of order preserving permutations of **Q** whose supports are right bounded (that is, $\bigvee f \in D\ \exists \alpha_f \in \mathbf{Q}$ such that $\forall \beta > \alpha_f$, $\beta f = \beta$) and consist of a finite number of linear pieces. We order D the same way as CH. Notice that CH is an o-subgroup of D. Also note that D is ordinally simple. For a description of more general o-groups of this type, see Dlab [4].

We now show that Clifford's o-group like Chehata's can be realized as an o-subgroup of D.

Theorem. CL *can be o-embedded into D.*

Proof. Let g_r, $r \in \mathbf{Q}$, be a generator of CL. Define $g_r\varphi \in D$ by

$$(\alpha)(g_r\varphi) = \begin{cases} \alpha & \text{if } \alpha \geqslant r, \\ \frac{1}{2}(\alpha + r) & \text{if } \alpha < r. \end{cases}$$

Extend φ to include $\{g_r^{-1}\}$, $r \in \mathbf{Q}$, so that $(g_r^{-1}\varphi) = (g_r\varphi)^{-1}$. That is,

$$(\alpha)(g_r^{-1}\varphi) = \begin{cases} \alpha & \text{if } \alpha \geqslant r, \\ 2\alpha - r & \text{if } \alpha < r. \end{cases}$$

A routine calculation shows that for $s < r$:

$$g_{(s+r)/2}\varphi = (g_r\varphi)^{-1}(g_s\varphi)(g_r\varphi). \tag{$*$}$$

So for an arbitrary element $g \in \text{CL}$ with normal form $g = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$, define $g\varphi = (g_{r_1}\varphi)^{n_1}(g_{r_2}\varphi)^{n_2} \cdots (g_{r_k}\varphi)^{n_k}$.

($*$) shows that this is a well-defined homomorphism.

φ is order-preserving for if $g = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$ is in normal form, then for $r_{k-1} < \alpha < r_k$,

$$(\alpha)(g\varphi) = \begin{cases} ((2^{n_k} - 1)r_k + \alpha)/2^{n_k} & \text{if } n_k > 0, \\ 2|n_k|\alpha - (2|n_k| - 1)r_k & \text{if } n_k < 0. \end{cases}$$

In either case this shows that the slope of the rightmost nonidentity piece is $(\frac{1}{2})^{n_k}$. So if $n_k > 0$ (i.e. $g > 0$), $g\varphi$ is a positive element of D. As a consequence φ is 1-1.

One should note that (CL)φ consists of those elements of D whose slopes of the linear pieces are integral powers of $\frac{1}{2}$. In fact, if $g = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$ the linear pieces break at $r_1, r_2, \ldots, r_k$ and the slope of the piece with domain $[r_{i-1}, r_i]$ is $(\frac{1}{2})^n$ where $n = \sum_{j=i}^{n} n_j$.

From now on we freely interchange CL and (CL)φ. Context will indicate which form of CL we use.

Let $B \subseteq \text{CL}$ be those elements of CL with bounded support. By the above remarks, it is clear that if $g \in B$ with $g = g_{r_1}^{n_1}g_{r_2}^{n_2} \cdots g_{r_k}^{n_k}$ then $\sum_{i=1}^{k} n_i = 0$. Chehata [2] has shown that the commutator subgroup of CL is precisely those elements whose sum of powers is 0. Hence $B \subseteq$ [CL, CL] and in fact it is a normal subgroup. Chehata [2] incorrectly states that the set of elements of [CL, CL] which have the sum of their positive powers divisible by a fixed n is a normal subgroup of [CL, CL]. But this is not a subgroup, for example if $n = 3$, $g_1^{-3}g_2^3$ and $g_2^{-1}g_3^{-2}g_4^3$ are elements of [CL, CL] but their product is $g_1^{-3}g_2^2g_3^{-2}g_4^3$. The group B is the only normal subgroup of [CL, CL] known to the author.

LEMMA (HIGMAN [5]). *If* (G, Ω) *is any permutation group that satisfies*:

> *Whenever* $f, g, h \in G$ *with* $h \neq e$ *then* $\exists k \in G$ *such that* $(\mathrm{supp}(f) \cup \mathrm{supp}(g))kh \cap (\mathrm{supp}(f) \cup \mathrm{supp}(g))k = \varnothing$, *then* $[G, G]$ *is simple.*

THEOREM. CL *contains a nonabelian simple o-subgroup. In fact, the commutator subgroup of* B *is simple.*

PROOF. Clearly B (and CL) is 0-2 transitive (that is, if $\alpha_1 < \beta_1$, and $\alpha_2 < \beta_2$ then $\exists g \in B$ such that $\alpha_1 g = \alpha_2$ and $\beta_1 g = \beta_2$). Let $f, g, h \in B$ and suppose $(\mathrm{supp}(f) \cup \mathrm{supp}(g)) \subseteq [\alpha, \beta]$. Pick $\gamma \in \mathrm{supp}(h)$. Either $\gamma h > \gamma$ or $\gamma h < \gamma$. Without loss of generality assume the former. By 0-2 transitivity $\exists k \in B$ such that $\alpha k = \gamma$ and $\beta k = \alpha h$. Then $\alpha kh = \gamma h = \beta k > (\mathrm{supp}(f) \cup \mathrm{supp}(g))k$ and so Higman's Lemma says that $[B, B]$ is simple.

Question. Is $[B, B] = B$? If it were, then of course the above theorem would be more appealing.

We can define other "Clifford-like" groups by adjusting the conjugation as follows. For $0 < t < 1$, define CL_t to be the o-group generated by $\{g_r\}$, $r \in \mathbf{Q}$, such that $g_r^{-1} g_s g_r = g_{(1-t)r+ts}$ for $r > s$. Order CL_t in the same manner as CL. Note that if $t = \frac{1}{2}$, we have Clifford's o-group. We can now embed CL_t into D in a completely analogous fashion. The generators of CL_t will then have t as the slope of their first nonidentity piece. The permutations determined by elements of CL_t will then be those of D that have as slopes of their linear pieces powers of t. The theorem concerning the simple subgroup of CL_t is of course the same as for CL.

REFERENCES

1. C. G. Chehata, *An algebraically simple ordered group*, Proc. London Math. Soc. (3) **2** (1952), 183–197.
2. ______, *On Clifford's ordinally simple group*, J. Natur. Sci. and Math. **15** (1975), 79–82.
3. A. H. Clifford, *A non-commutative ordinally simple group*, Proc. Amer. Math. Soc. **2** (1952), 902–903.
4. V. Dlab, *On a family of simple ordered groups*, J. Austral. Math. Soc. **8** (1968), 591–608.
5. G. Higman, *On infinite simple groups*, Publ. Math. Debrecen **3** (1954), 221–226.
6. B. H. Neumann, *On ordered groups*, Amer. J. Math. **71** (1949), 1–18.

DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403