PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, August 1980

SUBGROUPS OF $ax + b$ AND THE SPLITTING OF TRIANGULAR GROUP SCHEMES[1]

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ABSTRACT. The subgroup schemes of the $ax + b$ group are computed. This leads to a quick proof that a triangular group scheme over an algebraically closed field is a semidirect product of unipotent and diagonalizable subgroups.

The closed subgroup schemes of the $ax + b$ group (affine group of the line) do not seem to be worked out anywhere in print. I compute them here mainly because that turns out to be the only computation needed in a short new proof of a well-known splitting theorem for closed subgroups of the upper triangular group. The only previous proofs I know were given by A. Borel [1, p. 56], [2, p. 247] (treating the smooth case), M. Raynaud [4, pp. 573–601] (an extensive study of the result and its generalizations), and J. B. Sullivan [5] (a purely Hopf-algebraic argument). They all follow rather different lines, though with hindsight I can see traces of the $ax + b$ computation in Sullivan's argument. Corollary 2 below could also be proved a bit faster by Ext-group computations generalizing [3, p. 454], but I have avoided this to emphasize the basic simplicity of the result.

1. In any commutative ring the pairs (a, b) with a invertible become a group under the multiplication $(a, b)(c, d) = (ac, ad + b)$; this is an affine group scheme (see e.g. [3] for definitions). It is isomorphic to the group of coordinate changes $x \mapsto ax + b$, and we call it the $ax + b$ group. Projection to the first coordinate is a homomorphism onto the multiplicative group G_m with kernel the additive group G_a.

THEOREM 1. *Let k be a field of characteristic exponent p. Then the closed subgroups of the $ax + b$ group over k have the form*

$$H = \{(a, b) | a^n = 1, \varphi(b) = r(a^{p^i} - 1)\}.$$

Here n and i are nonnegative integers, $\varphi(X)$ is an additive polynomial $\sum s_j X^{p^j}$, and all p^j occurring are congruent to p^i modulo n.

PROOF. The only closed subgroups of G_m over k are the $\mu_n = \{a | a^n = 1\}$, where for convenience we let $n = 0$ stand for G_m itself. Given a closed subgroup H, we choose n describing the image of H in G_m. The only closed subgroups of G_a are the kernels $\varphi = 0$ of additive polynomials [3, p. 483], and we choose φ describing $H \cap G_a$.

Received by the editors July 6, 1978 and, in revised form, September 14, 1979.
AMS (MOS) subject classifications (1970). Primary 14L15, 20G15.
[1]This work was partially supported by the National Science Foundation.

Since $(a, b)(1, c)(a, b)^{-1} = (1, ac)$, we see that $\varphi(ac) = \Sigma s_j a^{p^j} c^{p^j}$ must vanish whenever $\varphi(c) = 0$ and $a^n = 1$. Taking $a = [Y]$ and $c = [Z]$ in the k-algebra $k[Y, Y^{-1}, Z]/(Y^n - 1, \varphi(Z))$, we find that the condition is satisfied only if all p^j are congruent modulo n. We choose some such p^j and call it p^i. This will not make sense if φ is identically zero, i.e. $G_a \subseteq H$; but in that case we have the full inverse image of μ_n, so we choose i at random and set $r = 0$.

For each given a, the b with (a, b) in H form a coset of $H \cap G_a$ defined by some equation φ = constant. Thus there is some function f: $\mu_n \to G_a$ defined over k with $H = \{(a, b) | a^n = 1, \varphi(b) = f(a)\}$. The condition for this to be closed under multiplication easily works out to be $f(aa') = f(a) + a^{p^i} f(a')$. In Hopf algebra terms, this means that f in $k[X, X^{-1}]/(X^n - 1)$ must satisfy $\Delta(f) = f \otimes 1 + X^{p^i} \otimes f$. It is trivial to check that this implies $f(a) = r(a^{p^i} - 1)$ for some constant r. □

COROLLARY 2. *Assume k is algebraically closed. Then every subgroup H is the semidirect product of $H \cap G_a$ and a group isomorphic to μ_n.*

PROOF. As k is algebraically closed, we can find some t in k with $\varphi(t) = r$. Then $\{(a, t(1 - a)) | a^n = 1\}$ is a subgroup of H complementary to $H \cap G_a$. □

2. One calls an affine algebraic group scheme G over a field k *triangular* if for some r it is isomorphic to a closed subgroup of the invertible upper triangular $r \times r$ matrices. This is equivalent to saying that all irreducible linear representations of G are one dimensional. (In Hopf algebra terms, the coradical is spanned by grouplike elements.) In any triangular embedding, the map of G to the diagonal gives a diagonalizable quotient with kernel the strictly upper triangular part $G \cap U$; this $G \cap U$ is the largest unipotent subgroup of G.

LEMMA 3. *Let G be triangular with $G \cap U$ nontrivial. Then there is a homomorphism from G to the $ax + b$ group which is nontrivial on $G \cap U$.*

PROOF. Let G be upper triangular in the basis $v_1, \ldots, v_r$. By assumption G is not diagonalizable, so there is an s such that G acts diagonalizably on $kv_1 + \cdots + kv_{s-1}$ but not on $kv_1 + \cdots + kv_s$. Changing the basis if necessary, we may assume $v_1, \ldots, v_{s-1}$ are eigenvectors for G, so that the matrix entries $x_{ij}(g)$ are zero when $i \neq j < s$. One of the x_{is} is nontrivial on $G \cap U$, and $x_{is}(gg') = x_{ii}(g)x_{is}(g') + x_{is}(g)x_{ss}(g')$. Then $a = x_{ss}^{-1}x_{ii}$ and $b = x_{ss}^{-1}x_{is}$ give the homomorphism. □

THEOREM 4. *Let G be triangular, and assume k is algebraically closed. Then G is the semidirect product of $G \cap U$ and a diagonalizable group.*

PROOF. Let T be minimal among closed subgroups mapping onto $G/G \cap U$. If $T \cap U$ is trivial, the theorem is proved. If not, there is a homomorphism φ, nontrivial on $T \cap U$, mapping T to the $ax + b$ group. We have $\varphi(T \cap U)$ unipotent, while $\varphi(T)/\varphi(T \cap U)$ is an image of $T/T \cap U$ and hence is diagonalizable; thus $\varphi(T \cap U)$ is precisely $\varphi(T) \cap G_a$. By Corollary 2 we can write $\varphi(T)$ as $\varphi(T \cap U) \cdot T_0$ for some diagonalizable group T_0. As φ is nontrivial on $T \cap U$, the group $\varphi^{-1}(T_0)$ is a proper closed subgroup of T. But $T = (T \cap U)\varphi^{-1}(T_0)$, so

$\varphi^{-1}(T_0)$ maps onto $T/T \cap U \simeq G/G \cap U$. This contradicts the minimality of T. □

One can drop the assumption that G is of finite type: as in [5], the theorem holds for all affine group schemes over k whose irreducible representations are one dimensional. Indeed, the lemma remains valid, because if G is not diagonalizable it has some nondiagonalizable quotient of finite type. In the proof of the theorem, the only point requiring attention is the existence of a minimal T. But if $T_\alpha = \mathrm{Spec}(k[G]/I_\alpha)$ is a decreasing chain of subgroups mapping onto $G/G \cap U$, then $\bigcap T_\alpha = \mathrm{Spec}(k[G]/\bigcup I_\alpha)$ also maps onto it, since by [3, p. 353] this means only that the Hopf subalgebra $k[G/G \cap U]$ has trivial intersection with $\bigcup I_\alpha$. Thus Zorn's lemma applies.

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