SUBGROUPS OF ax + b AND THE SPLITTING OF TRIANGULAR GROUP SCHEMES

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Abstract. The subgroup schemes of the ax + b group are computed. This leads to a quick proof that a triangular group scheme over an algebraically closed field is a semidirect product of unipotent and diagonalizable subgroups.

The closed subgroup schemes of the ax + b group (affine group of the line) do not seem to be worked out anywhere in print. I compute them here mainly because that turns out to be the only computation needed in a short new proof of a well-known splitting theorem for closed subgroups of the upper triangular group. The only previous proofs I know were given by A. Borel [1, p. 56], [2, p. 247] (treating the smooth case), M. Raynaud [4, pp. 573–601] (an extensive study of the result and its generalizations), and J. B. Sullivan [5] (a purely Hopf-algebraic argument). They all follow rather different lines, though with hindsight I can see traces of the ax + b computation in Sullivan’s argument. Corollary 2 below could also be proved a bit faster by Ext-group computations generalizing [3, p. 454], but I have avoided this to emphasize the basic simplicity of the result.

1. In any commutative ring the pairs (a, b) with a invertible become a group under the multiplication (a, b)(c, d) = (ac, ad + b); this is an affine group scheme (see e.g. [3] for definitions). It is isomorphic to the group of coordinate changes 

x \mapsto ax + b, and we call it the ax + b group. Projection to the first coordinate is a homomorphism onto the multiplicative group \( \mathbb{G}_m \) with kernel the additive group \( \mathbb{G}_a \).

Theorem 1. Let k be a field of characteristic exponent p. Then the closed subgroups of the ax + b group over k have the form

\[
H = \{(a, b) | a^n = 1, \varphi(b) = r(a^{n'} - 1)\}.
\]

Here \( n \) and \( i \) are nonnegative integers, \( \varphi(X) \) is an additive polynomial \( \sum s_j X^{p^j} \), and all \( p^j \) occurring are congruent to \( p^i \) modulo \( n \).

Proof. The only closed subgroups of \( \mathbb{G}_m \) over k are the \( \mu_n = \{a | a^n = 1\} \), where for convenience we let \( n = 0 \) stand for \( \mathbb{G}_m \) itself. Given a closed subgroup \( H \), we choose \( n \) describing the image of \( H \) in \( \mathbb{G}_m \). The only closed subgroups of \( \mathbb{G}_a \) are the kernels \( \varphi = 0 \) of additive polynomials [3, p. 483], and we choose \( \varphi \) describing \( H \cap \mathbb{G}_a \).
Since \((a, b)(1, c)(a, b)^{-1} = (1, ac)\), we see that \(\varphi(ac) = \sum s_j a^{s_j} c^{s_j}\) must vanish whenever \(\varphi(c) = 0\) and \(a^n = 1\). Taking \(a = [Y]\) and \(c = [Z]\) in the \(k\)-algebra \(k[Y, Y^{-1}, Z]/(Y^n - 1, \varphi(Z))\), we find that the condition is satisfied only if all \(p_j\) are congruent modulo \(n\). We choose some such \(p_j\) and call it \(p_i\). This will not make sense if \(\varphi\) is identically zero, i.e. \(G_a \subseteq H\); but in that case we have the full inverse image of \(\mu_n\) so we choose \(i\) at random and set \(r = 0\).

For each given \(a\), the \(b\) with \((a, b)\) in \(H\) form a coset of \(H \cap G_a\) defined by some equation \(\varphi = \) constant. Thus there is some function \(f: \mu_n \to G_a\) defined over \(k\) with \(H = \{(a, b)|a^n = 1, \varphi(b) = f(a)\}\). The condition for this to be closed under multiplication easily works out to be \(f(aa') = f(a) + a^n f(a')\). In Hopf algebra terms, this means that \(f\) in \(k[X, X^{-1}]/(X^n - 1)\) must satisfy \(\Delta(f) = f \otimes 1 + X^n f \otimes f\). It is trivial to check that this implies \(f(a) = r(a^n - 1)\) for some constant \(r\). □

**Corollary 2.** Assume \(k\) is algebraically closed. Then every subgroup \(H\) is the semidirect product of \(H \cap G_a\) and a group isomorphic to \(\mu_n\).

**Proof.** As \(k\) is algebraically closed, we can find some \(t\) in \(k\) with \(\varphi(t) = r\). Then \([(a, (1 - a))]|a^n = 1\) is a subgroup of \(H\) complementary to \(H \cap G_a\). □

2. One calls an affine algebraic group scheme \(G\) over a field \(k\) triangular if for some \(r\) it is isomorphic to a closed subgroup of the invertible upper triangular \(r \times r\) matrices. This is equivalent to saying that all irreducible linear representations of \(G\) are one dimensional. (In Hopf algebra terms, the coradical is spanned by group-like elements.) In any triangular embedding, the map of \(G\) to the diagonal gives a diagonalizable quotient with kernel the strictly upper triangular part \(G \cap U\); this \(G \cap U\) is the largest unipotent subgroup of \(G\).

**Lemma 3.** Let \(G\) be triangular with \(G \cap U\) nontrivial. Then there is a homomorphism from \(G\) to the \(ax + b\) group which is nontrivial on \(G \cap U\).

**Proof.** Let \(G\) be upper triangular in the basis \(v_1, \ldots, v_r\). By assumption \(G\) is not diagonalizable, so there is an \(s\) such that \(G\) acts diagonalizably on \(kv_1 + \cdots + kv_{s-1}\) but not on \(kv_1 + \cdots + kv_s\). Changing the basis if necessary, we may assume \(v_1, \ldots, v_{s-1}\) are eigenvectors for \(G\), so that the matrix entries \(x_{ij}(g)\) are zero when \(i \neq j < s\). One of the \(x_{is}\) is nontrivial on \(G \cap U\), and \(x_{is}(gg') = x_{is}(g)x_{is}(g') + x_{is}(g)x_{is}(g')\). Then \(a = x_{is}^{-1}x_{is}\) and \(b = x_{is}^{-1}x_{is}\) give the homomorphism. □

**Theorem 4.** Let \(G\) be triangular, and assume \(k\) is algebraically closed. Then \(G\) is the semidirect product of \(G \cap U\) and a diagonalizable group.

**Proof.** Let \(T\) be minimal among closed subgroups mapping onto \(G/G \cap U\). If \(T \cap U\) is trivial, the theorem is proved. If not, there is a homomorphism \(\varphi\), nontrivial on \(T \cap U\), mapping \(T\) to the \(ax + b\) group. We have \(\varphi(T \cap U)\) unipotent, while \(\varphi(T)/\varphi(T \cap U)\) is an image of \(T/T \cap U\) and hence is diagonalizable; thus \(\varphi(T \cap U)\) is precisely \(\varphi(T) \cap G_a\). By Corollary 2 we can write \(\varphi(T)\) as \(\varphi(T \cap U) \cdot T_0\) for some diagonalizable group \(T_0\). As \(\varphi\) is nontrivial on \(T \cap U\), the group \(\varphi^{-1}(T_0)\) is a proper closed subgroup of \(T\). But \(T = (T \cap U)\varphi^{-1}(T_0)\), so
\( \varphi^{-1}(T_\alpha) \) maps onto \( T/T \cap U \cong G/G \cap U \). This contradicts the minimality of \( T \).

One can drop the assumption that \( G \) is of finite type: as in [5], the theorem holds for all affine group schemes over \( k \) whose irreducible representations are one dimensional. Indeed, the lemma remains valid, because if \( G \) is not diagonalizable it has some nondiagonalizable quotient of finite type. In the proof of the theorem, the only point requiring attention is the existence of a minimal \( T \). But if \( T_\alpha = \text{Spec}(k[G]/I_\alpha) \) is a decreasing chain of subgroups mapping onto \( G/G \cap U \), then \( \cap T_\alpha = \text{Spec}(k[G]/\cup I_\alpha) \) also maps onto it, since by [3, p. 353] this means only that the Hopf subalgebra \( k[G/G \cap U] \) has trivial intersection with \( \cup I_\alpha \). Thus Zorn’s lemma applies.

**References**


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