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FROBENIUS EXTENSIONS OF *QF*-3 RINGS

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ABSTRACT. We investigate the inheritance of *QF*-3 property for ring extensions, mainly, for Frobenius extensions. Let A be a ring with identity. It is proved that a group ring $A[G]$ of A with a finite group G is left *QF*-3 iff A is left *QF*-3 and that in case A is a G-Galois extension of the fixed subring A^G relative to a finite group G of ring automorphism of A, A is left *QF*-3 iff A^G is left *QF*-3.

Let A be a ring with identity. It is well know that a group ring $A[G]$ with a finite group G is Quasi-Frobenius (*QF*) iff A is *QF*. Using the concept of Frobenius extensions introduced by F. Kasch [4], we shall obtain a similar result for *QF*-3 rings in this paper, namely, $A[G]$ is left *QF*-3 iff A is left *QF*-3. Here a ring is called left *QF*-3 if it has a minimal faithful left module, that is, a faithful left module which is isomorphic to a direct summand of every faithful left module. Further we shall show that in case A is a G-Galois extension of the fixed subring A^G relative to a finite group G of ring automorphism of A in the sense of [7], A is left *QF*-3 iff A^G is left *QF*-3. It should be noted that A and A^G are not always left *QF*-3 even if A^G and A are left *QF*-3, respectively and that in case A/A^G is finite G-Galois, A is *QF* whenever A^G is *QF* but the converse is not necessarily true.

Throughout this paper, all rings, all modules, all subrings and all ring homomorphisms are assumed to be unitary. We follow the notation of [10] unless specified otherwise. For A-A'-bimodules ${}_AM_{A'}$, ${}_AN_{A'}$ the notation ${}_AM_{A'}|{}_AN_{A'}$ denotes the fact that ${}_AM_{A'}$ is isomorphic to a direct summand of a direct sum $N^{(n)}$ of finitely many copies of ${}_AN_{A'}$. A module M is said to be *cofinitely generated* (co-f.g.) in case for every set $\{M_i;\ i \in I\}$ of submodules of M if the intersection $\bigcap_i M_i = 0$, then there exist $i_1, \ldots, i_n$ in I such that $\bigcap_k M_{i_k} = 0$ (see [11]). If a module M is f.g. projective, co-f.g. injective and faithful, then M will be called a $*$-module for convenience. If a ring A is left *QF*-3, then a minimal faithful left A-module is a $*$-module, and conversely if A has a left $*$-module, then A is left *QF*-3 (see [2, Theorem 1]). Let $A \supset B \ni 1_A$ be a ring extension. We say A is a *Frobenius* (resp. a *left QF*) *extension* of B if ${}_BA$ is f.g. projective and if ${}_AA_B \cong$ (resp. $|$) ${}_A\mathrm{Hom}({}_BA, {}_BB)_B$, and a right *QF* extension is defined symmetrically (see [4] and [8]). If A is a Frobenius extension of B, then there exist a B-B-homomorphism h of A to B and $r_1, \ldots, r_n;\ l_1, \ldots, l_n$ in A such that $x = \sum_i r_i h(l_i x) = \sum_i h(x r_i) l_i$ for all x in A, and conversely (see [9]). When this is the case, we shall call such a system $(h;\ l_i, r_i)_{1 < i < n}$ a *Frobenius system*.

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PROPOSITION 1. *Let $A \supset B$ be a ring extension. Let Y be a faithful left B-module. If ${}_BA$ (resp. A_B) is torsionless, then ${}_A\mathrm{Hom}({}_BA, {}_BY)$ (resp. ${}_AA \otimes {}_BY$) is faithful.*

PROOF. Let a be an arbitrary nonzero element of A. First assume that ${}_BA$ is torsionless. Then there is f: ${}_BA \to {}_BB$ such that $(a)f \neq 0$. But, ${}_BY$ being faithful, we have $(a)f \cdot y \neq 0$ for some y in Y, and so $(1)(a \cdot (f \circ g_y)) = (a)f \cdot y \neq 0$, where g_y: ${}_BB \to {}_BY$ is given by $(b)g_y = by$ for b in B. Thus ${}_A\mathrm{Hom}({}_BA, {}_BY)$ is faithful. Next assume that A_B is torsionless. Then there is f: $A_B \to B_B$ such that $f(a) \neq 0$. But, ${}_BY$ being faithful, we have $f(a) \cdot y \neq 0$ for some y in Y, and so $a(1 \otimes y) = a \otimes y \neq 0$ in $A \otimes {}_BY$. Thus ${}_AA \otimes {}_BY$ is faithful.

The following is well known (see e.g. [1, Exercise 10, p. 261]).

LEMMA 2. *Let $A \supset B$ be a ring extension. Then the following hold.*

(1) *If A_B is flat and ${}_AX$ is injective, then ${}_BX$ is injective.*

(2) *If ${}_BA$ is f.g. projective, and if ${}_AX$ is f.g. projective, then ${}_BX$ is f.g. projective.*

PROPOSITION 3. *Let $A \supset B$ be a ring extension. Then the following hold.*

(1) *If A_B is torsionless and if ${}_AA$ is co-f.g., then ${}_BB$ is co-f.g.*

(2) *If ${}_BA$ is imbedded in a direct sum $B^{(n)}$ of finitely many copies of B, and if ${}_BB$ is co-f.g., then ${}_AA$ is co-f.g.*

PROOF. (1) Let $\{X_i;\ i \in I\}$ be a set of submodules of ${}_BB$ such that $\bigcap_i X_i = 0$. Suppose that $a \in AX_i$ for all $i \in I$. Then, for any f: $A_B \to B_B$, we have $f(a) \in X_i$ for all $i \in I$, and so $f(a) = 0$. Thus, A_B being torsionless, we have $a = 0$, that is, $\bigcap_i AX_i = 0$. Since ${}_AA$ is co-f.g., there are $i_1, \ldots, i_s \in I$ such that $\bigcap_k AX_{i_k} = 0$, and so $\bigcap_k X_{i_k} = 0$, proving (1). (2) is evident.

PROPOSITION 4 (cf. [5, Theorem 2.4]). *Let $A \supset B$ be a ring extension such that A_B and ${}_BA$ are f.g. projective. If A is left QF-3 such that ${}_AA$ is co-f.g., then B is left QF-3 such that ${}_BB$ is co-f.g. The converse is true if $A \supset B$ is a left or right QF extension.*

PROOF. Suppose that A is left QF-3 such that ${}_AA$ is co-f.g. Let ${}_AU$ be a *-module. We shall show that U is a *-module as a left B-module. By Lemma 2, ${}_BU$ is f.g. projective and injective. On the other hand, ${}_BB$ is co-f.g. by Proposition 3. Thus ${}_BU$ is co-f.g. Further ${}_BU$ is faithful obviously. Hence we obtain the first half. Conversely assume that B is left QF-3 such that ${}_BB$ is co-f.g. Then ${}_AA$ is co-f.g. by Proposition 3. Let ${}_BV$ be a *-module. It is obvious that ${}_AA \otimes {}_BV$ is f.g. projective and that ${}_A\mathrm{Hom}({}_BA, {}_BV)$ is injective. Thus the first module is co-f.g. Further both modules are faithful by Proposition 1. If $A \supset B$ is a left QF extension, then we have ${}_AA \otimes {}_BV | {}_A\mathrm{Hom}({}_BA, {}_BB) \otimes {}_BV \cong {}_A\mathrm{Hom}({}_BA, {}_BV)$, and so ${}_AA \otimes {}_BV$ is injective. If $A \supset B$ is a right QF extension, then we have ${}_A\mathrm{Hom}({}_BA, {}_BV) | {}_AA \otimes {}_BV$. Thus, ${}_A\mathrm{Hom}({}_BA, {}_BV)$ is f.g. projective, and so it is co-f.g. It follows that ${}_AA \otimes {}_BV$ is a *-module in the first case and that ${}_A\mathrm{Hom}({}_BA, {}_BV)$ is a *-module in the second case. Thus the proof is complete.

REMARK. If $A \supset B$ is a left or right QF extension, then ${}_BA$ and A_B are f.g. projective.

Following [6], a bimodule ${}_AM_A$ is said to be *generated by normalizing elements* if there are sets $\{m_i; i \in I\} \subset M$ and $\{\sigma_i; i \in I\} \subset \mathrm{Aut}(A)$ such that $M = \sum_i Am_i$ and $m_i a = \sigma_i(a)m_i$ for all $i \in I$, $a \in A$, where $\mathrm{Aut}(A)$ denotes the set consisting of all ring automorphisms of A. Let ${}_AX$ be an arbitrary A-module and σ an arbitrary ring endomorphism of A. Then a left A-module X_σ is defined as follows. X_σ coincides with X as the additive group and the ring A operates on X_σ by $a \circ x = \sigma(a)x$ ($a \in A$, $x \in X_\sigma$).

PROPOSITION 5. *Let $A \supset B$ be a ring extension. Suppose that A is f.g. over B by normalizing elements, and let $\{a_1, \ldots, a_t\} \subset A$, $\{\sigma_1, \ldots, \sigma_t\} \subset \mathrm{Aut}(B)$ be subsets such that $A = \sum_i Ba_i$ and $a_i b = \sigma_i(b)a_i$ for all i and $b \in B$. Let X be an arbitrary left B-module. Then the mapping*

$$\eta_X\colon \mathrm{Hom}({}_BA, {}_BX) \to \bigoplus_{i=1}^{t} X_{\sigma_i}, \quad f \mapsto (a_i f)_i,$$

is a left B-monomorphism. Consequently, if ${}_BX$ is co-f.g., then ${}_A\mathrm{Hom}({}_BA, {}_BX)$ is co-f.g. as a B-module and hence as an A-module.

PROOF. It is easy to see that η_X is a B-monomorphism. If ${}_BX$ is co-f.g., then obviously so is X_{σ_i} for every σ_i. Thus, η_X being B-monic, we obtain the second assertion.

PROPOSITION 6. *Let $A \supset B$ be a ring extension such that ${}_BA$ and A_B are f.g. projective. Suppose that A is f.g. over B by normalizing elements. If A is left QF-3, then B is left QF-3. The converse is true if $A \supset B$ is a left or right QF extension.*

PROOF. Suppose that A is left QF-3 and let ${}_AU$ be a minimal faithful module. Then ${}_BU$ is f.g. projective, injective and faithful as mentioned previously. To prove ${}_BU$ co-f.g., let $\{S_i; i \in I\}$ be a complete set of representatives for the distinct isomorphism classes of simple left B-modules. Set $X = \bigoplus_i E(S_i)$, where $E(S_i)$ denotes the injective envelope of S_i. Since ${}_BX$ is a cogenerator, ${}_A\mathrm{Hom}({}_BA, {}_BX)$ is a cogenerator as is easily seen. But, ${}_BA$ being f.g., the last module is isomorphic to $\bigoplus_i \mathrm{Hom}({}_BA, {}_BE(S_i))$. Hence, recalling ${}_AU$ f.g., there are $i_1, \ldots, i_s \in I$ such that ${}_AU$ can be imbedded in $\bigoplus_{k=1}^{s}\mathrm{Hom}({}_BA, {}_BE(S_{i_k}))$ which is co-f.g. as a B-module by Proposition 5. Thus ${}_BU$ is co-f.g. It follows that B is left QF-3. Conversely assume that B is left QF-3. Let ${}_BV$ be a $*$-module. Then ${}_A\mathrm{Hom}({}_BA, {}_BV)$ is co-f.g. by Proposition 5. Moreover we can see by the same way as the proof of Proposition 4 that ${}_AA \otimes {}_BV$ (resp. ${}_A\mathrm{Hom}({}_BA, {}_BV)$) is a $*$-module if $A \supset B$ is a left (resp. right) QF extension. It follows that A is left QF-3.

As an immediate consequence of Proposition 6, we have

COROLLARY 1. *Let $A \supset B$ be a left or right QF extension such that A is f.g. as a B-module by elements which commute with every element of B. Then A is left QF-3 iff B is left QF-3.*

COROLLARY 2. *A group ring $A[G]$ of a ring A with a finite group G is left QF-3 iff A is left QF-3.*

PROOF. It is easy to see that $(h; \sigma, \sigma^{-1})_{\sigma\in G}$ is a Frobenius system for $A[G]/A$, where $h: A[G] \to A$ is defined by $h(\Sigma_\sigma a_\sigma \cdot \sigma) = a_1$. Thus Corollary 2 follows from Corollary 1.

We can also have the following well-known result as a consequence of Proposition 6.

COROLLARY 3. *$(A)_n$ is left QF-3 iff A is left QF-3.*

PROOF. It is easy to see that $(\mathrm{tr}; E_{ij}, E_{ji})_{i,j}$ is a Frobenius system for $(A)_n \supset A$, where $\mathrm{tr}: (A)_n \to A$ is defined by $\mathrm{tr}((a_{ij})) = \Sigma_i a_{ii}$, and E_{ij} $(1 \leqslant i, j \leqslant n)$ denotes the matrix in $(A)_n$ with 1 in the (i, j)-component and 0 elsewhere. Thus Corollary 3 follows from Corollary 1.

PROPOSITION 7. *Let e be an idempotent of A such that ${}_AAe$ and eA_A are faithful. If A is left QF-3, then so is eAe.*

PROOF. Let Ae_1 $(e_1^2 = e_1 \in A)$ be a unique minimal faithful left A-module, and set $V = eAe_1$ and $B = eAe$. Since ${}_AAe$ is faithful, ${}_AAe = {}_AAe_1 \oplus *$, and so ${}_BB \cong {}_B\mathrm{Hom}({}_AAe, {}_AAe) \cong {}_B\mathrm{Hom}({}_AAe, {}_AAe_1) \oplus ** \cong {}_BV \oplus **$. Thus ${}_BV$ is f.g. projective. Further ${}_BV$ is injective: Let L be a left ideal of B and f a homomorphism of L to V. Noting eA_A is faithful, it is not hard to see that a mapping $\bar{f}: AL \to Ae_1$ defined by $(\Sigma a_i x_i)\bar{f} = \Sigma a_i x_i f$ $(a_i \in A, x_i \in L)$ is well defined. Thus, Ae_1 being injective, there exists some $x_1 \in Ae_1$ such that $x\bar{f} = xx_1$ for all $x \in AL$. Hence we have $ex_1 \in V$ and $l\bar{f} = lf = lx_1 = lex_1$ for all $l \in L$. Therefore, ${}_BV$ is injective. To prove ${}_BV$ co-f.g., let $\{V_i; i \in I\}$ be a set of submodules of ${}_BV$ such that $\bigcap_i V_i = 0$. Suppose that $a \in AV_i$ for all $i \in I$. Since $eAa \subset eAAV_i = eAeV_i = V_i$ for all $i \in I$, we then have $eAa = 0$. But, eA_A being faithful, we have $a = 0$, that is, $\bigcap_i AV_i = 0$. Since Ae_1 is co-f.g., there are $i_1, \ldots, i_s \in I$ such that $\bigcap_k AV_{i_k} = 0$, which yields $\bigcap_k V_{i_k} = 0$. Hence ${}_BV$ is co-f.g. Finally, ${}_AAe_1$ being faithful, ${}_BV$ is faithful. It follows that B is left QF-3.

PROPOSITION 8. *Let $A \supset B$ be a Frobenius extension with Frobenius system $(h; l_i, r_i)_{1 \leqslant i \leqslant n}$. Then the following hold.*

(1) *$H = (h(l_i r_j)) \in (B)_n$ is an idempotent such that the left annihilator of $(B)_n H$ in $(B)_n$ and the right annihilator of $H(B)_n$ in $(B)_n$ both vanish.*

(2) $\mathrm{End}(A_B)$, *the endomorphism ring of A_B, is left QF-3 if B is left QF-3.*

PROOF. (1) Recalling $x = \Sigma_i r_i h(l_i x)$ for all $x \in A$, it is easy to see that H is an idempotent. Let E_{ij} $(1 \leqslant i, j \leqslant n)$ be the matrix with 1 in the (i, j)-component and 0 elsewhere. Let $Y = (b_{ij})$ be an arbitrary nonzero element of $(B)_n$, say $b_{st} \neq 0$. Since $1 = \Sigma h(r_i) l_i$, we have $b_{st} h(r_i) \neq 0$ for some i. Setting $Y' = \Sigma_j h(r_j) E_{tj}$, the (s, i)-component of $YY'H$ is equal to $b_{st} h(r_i) \neq 0$, which proves that the left annihilator of $(B)_n H$ in $(B)_n$ is zero. A similar argument shows that the right annihilator of $H(B)_n$ in $(B)_n$ is zero. (2) Recalling the mention at the begin. ing of the proof, the ring $\mathrm{End}(A_B)$ is isomorphic to $H(B)_n H$. Thus (2) follows from Proposition 7 together with (1) and Corollary 3 to Proposition 6.

Let G be a finite group of ring automorphism of A. Let A^G denote the fixed subring of A under G, i.e. $A^G = \{a \in A; \sigma(a) = a$ for all $\sigma \in G\}$. Let $\Delta = \Delta(A; G)$ be the trivial crossed product of A with G, that is, Δ is a free (left) A-module with free generator $\{u_\sigma\}$ indexed by G and with multiplication defined by $au_\sigma bu_\tau = a\sigma(b)u_{\sigma\tau}$. The ring Δ has u_1 for its identity and the mapping $a \mapsto au_1$ imbeds A as a subring of Δ. Moreover Δ is a Frobenius extension of A with Frobenius system $(h; u_\sigma, u_\sigma^{-1})_{\sigma \in G}$, where h is defined by $h(\Sigma_\sigma a_\sigma u_\sigma) = a_1 u_1$. Furthermore A has a natural structure as a left Δ-module by means of the operation $au_\sigma \circ x = a\sigma(x)$. The endomorphism ring $\mathrm{End}({}_\Delta A)$ then may be identified with A^G by the mapping $g \mapsto 1g$. In what follows, Δ will denote the trivial crossed product $\Delta(A; G)$ of A with G.

As an immediate consequence of Proposition 6, we have

PROPOSITION 9. *If A is left QF-3, then so is Δ, and conversely.*

Assume now that A is a G-Galois extension of A^G. Let $x_1, \ldots, x_n; y_1, \ldots, y_n$ be elements in A such that $\Sigma_i x_i \sigma(y_i) = \delta_{\sigma,1}$ for every $\sigma \in G$. Let $\mathrm{tr}(a)$ denote the trace of $a \in A$ defined by $\mathrm{tr}(a) = \Sigma_\sigma \sigma(a)$. Set $B = A^G$ and $C = \mathrm{End}(A_B)$. Then, as is easily seen, $(\mathrm{tr}; y_i, x_i)_{1 < i < n}$ is a Frobenius system for A/B. Moreover, a mapping j: $\Delta \to C$ defined by $j(au_\sigma)(x) = a\sigma(x)$ is a ring isomorphism whose inverse is given by $j^{-1}(f) = \Sigma_\sigma(\Sigma_i f(x_i)\sigma(y_i))u_\sigma$, and so, we shall identify C with Δ. If B is left QF-3, then Δ is left QF-3 by Proposition 8(2). Thus A is left QF-3 by Proposition 9. Conversely, if A is left QF-3, then Δ is left QF-3 by Proposition 9. Let ${}_\Delta U$ be a $*$-module, and set ${}_B V = {}_B\mathrm{Hom}({}_\Delta A, {}_\Delta U)$. We shall show ${}_B V$ is a $*$-module. It follows that B is left QF-3. Since A_B is f.g. projective, ${}_\Delta A$ is a generator as is well known. Further, the mapping $a \mapsto a\Sigma_\sigma u_\sigma$ imbedding A as a Δ-submodule of Δ, ${}_\Delta A$ is torsionless. Thus ${}_B V$ is f.g. projective, injective and faithful by [5, Proposition 2.5]. To prove ${}_B V$ co-f.g., let $\{V_i; i \in I\}$ be a chain of nonzero submodules of V. Then, recalling A_B f.g. projective, $\{A \otimes {}_B V_i; i \in I\}$ may be regarded as that of ${}_\Delta A \otimes {}_B V$. But ${}_\Delta A \otimes {}_B V$ is isomorphic to ${}_\Delta U$ by the mapping $a \otimes v \mapsto av$, because ${}_\Delta A$ is a generator. Thus, ${}_\Delta U$ being co-f.g., there exists some $x \in A \otimes {}_B V$ such that $0 \neq x \in A \otimes {}_B V_i$ for all $i \in I$. Since the mapping λ: $A \otimes {}_B V \to \mathrm{Hom}({}_B A, {}_B V)$ defined by $(a')(a \otimes v)\lambda = \mathrm{tr}(a'a)v$ is an isomorphism whose inverse λ^{-1}: $\mathrm{Hom}({}_B A, {}_B V) \to A \otimes {}_B V$ is given by $\lambda^{-1}(f) = \Sigma_i x_i \otimes y_i f$, we have $a \in A$ such that $(a)((x)\lambda)$ is nonzero and contained in V_i for all $i \in I$, which proves ${}_B V$ co-f.g. (see [1, Exercise 6, p. 131]).

We have proved the following.

THEOREM 10. *Assume that A is a G-Galois extension of A^G. Then A is left QF-3 iff A^G is left QF-3.*

REMARK. In the above theorem, the assumption seems to be very strong. The following examples illustrate that some hypothesis is needed about the relationship between A and A^G.

EXAMPLE 1. Let A be the subring

$$\begin{bmatrix} \mathbf{Q} & 0 & 0 \\ \mathbf{Q} & \mathbf{Z} & 0 \\ \mathbf{Q} & \mathbf{Q} & \mathbf{Q} \end{bmatrix}$$

of the 3×3 matrix ring $(\mathbf{Q})_3$, where $\mathbf{Q}$ denotes the field of rational numbers and $\mathbf{Z}$ the ring of integers. Let σ be the inner automorphism of A determined by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and $G = \langle \sigma \rangle$. Then A^G coincides with

$$\begin{bmatrix} \mathbf{Q} & 0 & 0 \\ \mathbf{Q} & \mathbf{Z} & 0 \\ 0 & 0 & \mathbf{Q} \end{bmatrix}.$$

As was mentioned in H. Tachikawa [**10**, pp. 44 and 70], A is left QF-3 and right QF-3 but A^G is neither left QF-3 nor right QF-3.

EXAMPLE 2. Let A be the subring of the 2×2 matrix ring $(\mathbf{R})_2$ consisting of all elements of the form $\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)$; $x \in \mathbf{Q}, y \in \mathbf{R}$, where $\mathbf{R}$ denotes the field of real numbers. Then A is a commutative ring without idempotents other than 0 and 1. If I denotes the ideal of A generated by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then $I = \left(\begin{smallmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{smallmatrix}\right)$ and $\operatorname{Ann}(\operatorname{Ann} I) = \left(\begin{smallmatrix} 0 & 0 \\ \mathbf{R} & 0 \end{smallmatrix}\right)$, where $\operatorname{Ann} I = \{a \in A;\ aI = 0\}$. Hence A is not a self-injective ring (see [**3**, Theorem 1]). Thus A is not QF-3. Let σ be the automorphism of A given by $\sigma\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ -y & x \end{smallmatrix}\right)$, and $G = \langle \sigma \rangle$. It is easy to see that A^G coincides with the field consisting of all elements of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$; $x \in \mathbf{Q}$.

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