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LOCAL ALGEBRAICITY OF SOME ANALYTIC HYPERSURFACES

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ABSTRACT. It is proved that an analytic hypersurface germ $(X, 0) \subseteq (\mathbf{C}^{n+1}, 0)$, with nonsingular normalization, whose only singularities outside the origin are normal crossings of two n-manifolds is isomorphic to a germ of an algebraic variety at 0. As a corollary we find that weakly normal surfaces $V \subseteq \mathbf{C}^3$ with nonsingular normalization are locally algebraic.

1. Introduction. It was proved by Samuel [7] that complex analytic hypersurface germs with an isolated singularity are analytically equivalent to an algebraic hypersurface in a neighborhood of the singularity. This result is an easy corollary of Mather's theorem on finitely determined function germs f: $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ (see e.g. Orlik [6]); following the approach of applying Mather's theorem, but applying the theorem to map germs rather than to function germs we can prove an analogue of the theorem of Samuel for a class of analytic hypersurface germs $(X, 0) \subseteq (\mathbf{C}^{n+1}, 0)$ with particularly simple codimension one singularities. We will begin by establishing notation and recalling the theorem of Mather [4].

All complex spaces are assumed to be reduced and pure dimensional. If $X \subseteq \mathbf{C}^{n+1}$ $(n > 2)$ is a complex analytic hypersurface, then we let $X_c = \{x \in X$: $\mathcal{O}_{X,x} \simeq \mathcal{O}_{\mathbf{C}^n,0}$ or $\mathcal{O}_{X,x} \simeq \mathcal{O}_{\{z_1z_2=0\}}, 0\}$. Thus X_c is open and $X \setminus X_c$ is an analytic subset of X. We will be considering those analytic hypersurface germs $(X, 0) \subseteq (\mathbf{C}^{n+1}, 0)$ such that $(X \setminus X_c, 0) \subseteq \{0\}$. We note in particular that such germs are weakly normal singularities, i.e., the sheaf of germs of continuously holomorphic functions is identical with the sheaf of germs of holomorphic functions [1].

Now let f: $N \to P$ be a holomorphic map of complex manifolds and let $S \subseteq N$ be a finite set such that $f(S) = y \in P$. We say that the map germ f_S is finitely determined if there is an integer k such that any other germ g_S, $g(S) = y$, with the same k-jet as f at S is analytically equivalent to f_S. To state the theorem of Mather we need the following notation.

Let $\theta(f)$ denote the sheaf of germs of holomorphic vector fields along f, let tf: $\theta(1_N) \to \theta(f)$ be the tangent map and let ωf: $\theta(1_P) \to \theta(f)$ be defined by $\omega f(\eta) = \eta \circ f$.

1.1. THEOREM (MATHER [4]). *Let $S \subseteq N$ be a finite set such that $f(S) = y \in P$. Then the germ f_S is finitely determined if and only if*

$$\dim_{\mathbf{C}} \theta(f)_S/(tf(\theta(1_N)_S) + \omega f(\theta(1_P)_y)) < \infty.$$

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2. Our result is the following:

2.1. THEOREM. *Let* $(V, 0) \subseteq (\mathbf{C}^{n+1}, 0)$ *be the germ at* 0 *of an analytic hypersurface, let* $f: W \to V$ *be the normalization of* V (*in a neighborhood of* 0), *and assume that*

(1) $(W, f^{-1}(0))$ *is nonsingular,*

(2) $V \setminus V_c \subseteq \{0\}$.

Then $(V, 0)$ *is analytically isomorphic to an algebraic hypersurface* $(Z, 0) \subseteq (\mathbf{C}^{n+1}, 0)$.

PROOF. Let $S = f^{-1}(0)$. We will show that the map germ $f_S: (W, S) \to (V, 0)$ is finitely determined. We will then have a commutative diagram

$$\begin{array}{ccc} (W, S) & \xrightarrow{f} & (\mathbf{C}^{n+1}, 0) \\ \zeta\downarrow & & \downarrow\psi \\ (W, S) & \xrightarrow{f^{(k)}} & (\mathbf{C}^{n+1}, 0) \end{array}$$

where ζ and ψ are analytic isomorphisms and $f^{(k)}$ is the kth order Taylor polynomial of f at S. Thus $V = f(W)$ is analytically equivalent near 0 to $f^{(k)}(W, S)$ which is algebraic.

To show that f_S is finitely determined we apply the criterion of Mather. Assume that a small enough neighborhood of S is chosen so that W is nonsingular and consists of $\# S$ connected components. If we identify $\theta(f)$ with the sheaf of germs of holomophic sections of $f^*T\mathbf{C}^{n+1}$, then $tf: \theta(1_W) \to \theta(f)$ is a map of $\mathcal{O}_W$-modules and we let $\mathfrak{F} = \operatorname{coker} tf$. This is a coherent sheaf of $\mathcal{O}_W$-modules and we let $\pi: \theta(f) \to \mathfrak{F}$ be the natural projection.

Let $\omega f: \mathcal{O}(1_{\mathbf{C}^{n+1}}) \to \theta(f)$ be the sheaf map defined by $\omega f(\eta) = \eta \circ f$. Note that this is a map of $\mathcal{O}_{\mathbf{C}^{n+1}}$-modules. By the theorem of Grauert **[5]** $f_*\mathfrak{F}$ is a coherent sheaf of $\mathcal{O}_{\mathbf{C}^{n+1}}$-modules; then $\pi \circ \omega f: \theta(1_{\mathbf{C}^{n+1}}) \to f_*\mathfrak{F}$ is a module map. Let $\mathcal{G} = \operatorname{Im}(\pi \circ \omega f)$ and $\mathcal{H} = (f_*\mathfrak{F})/\mathcal{G}$. Then $\mathcal{G}$ and hence $\mathcal{H}$ are coherent sheaves of $\mathcal{O}_{\mathbf{C}^{n+1}}$-modules and the stalk at 0 is

$$\mathcal{H}_0 = \theta(f)_S / \left(tf(\theta(1_W)_S) + \omega f(\theta(1_{\mathbf{C}^{n+1}})_0)\right).$$

Consider the sheaf of ideals of $\mathcal{O}_{n+1}$ defined by

$$\left(\mathcal{G} : f_*\mathfrak{F}\right)_y = \left\{ g \in \mathcal{O}_{\mathbf{C}^{n+1},y} : g \cdot f_*\mathfrak{F}_y \subseteq \mathcal{G}_y \right\}.$$

This is a coherent sheaf of ideals and since assumption (2) implies that f is an immersion with normal crossings outside of S, it follows that $\operatorname{Supp}(\mathcal{G} : f_*\mathfrak{F}) \subseteq \{0\}$ (because immersions with normal crossings are stable map germs). Then by the nullstellensatz $\mathfrak{M}_0^r \subseteq (\mathcal{G} : f_*\mathfrak{F})_0$ for some r where $\mathfrak{M}_0 \hookrightarrow \mathcal{O}_{n+1}$ is the maximal ideal. Thus $\mathfrak{M}_0^r \mathcal{H}_0 = 0$ so that $\dim_{\mathbf{C}} \mathcal{H}_0 < \infty$. Therefore Mather's criterion applies to show that f_S is finitely determined and the proof is complete.

2.2. COROLLARY. *A weakly normal surface* $(V, 0) \subseteq (\mathbf{C}^3, 0)$ *with nonsingular normalization is locally algebraic.*

PROOF. By the Oka type theorem for weakly normal complex spaces **[1]**, a surface $(V, 0) \subseteq (\mathbf{C}^3, 0)$ is weakly normal if and only if $V \setminus V_c \subseteq \{0\}$. Thus we may apply the theorem.

2.3. REMARKS. (1) In fact it is true that an analytic surface $(V, 0) \subseteq (\mathbf{C}^3, 0)$ with nonsingular normalization is weakly normal if and only if the normalization map f: $W \to V$ is finitely determined at $S = f^{-1}(0)$. Indeed, this follows from a continuation of the argument of the main theorem. What is needed is the result of Mather that a finitely determined germ f_S is stable outside of S. See the thesis of Gaffney [3] for the proof.

(2) For surface germs $(V, 0) \subseteq (\mathbf{C}^n, 0)$ $(n > 3)$ with nonsingular normalization, the equivalence between weak normality of $(V, 0)$ and the finite determination of the normalization map is no longer true. For example, let f: $\mathbf{C}^2 \to \mathbf{C}^4$ be the map $f(u, v) = (u, uv, v^2, v^3)$ and let $V = f(\mathbf{C}^2)$. Then f is finitely determined at 0 since it is a bijective immersion outside of 0, but $(V, 0)$ is not weakly normal [2].

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