PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, August 1980

CONVEX DOMAINS AND KOBAYASHI HYPERBOLICITY

THEODORE J. BARTH

ABSTRACT. A geometrically convex domain in $\mathbf{C}^n$ is Kobayashi hyperbolic if and only if it contains no complex affine lines. This contrasts with an example of a nonhyperbolic pseudoconvex domain in $\mathbf{C}^2$ containing no (nonconstant) entire holomorphic curves.

Recall that a complex space M is said to be hyperbolic if its Kobayashi pseudodistance d_M [4, pp. 97–98] is a distance. Since $d_{\mathbf{C}} = 0$ and holomorphic maps do not increase Kobayashi pseudodistances, every entire holomorphic curve $f: \mathbf{C} \to M$ in a hyperbolic complex space is constant. R. Brody [1, Theorem 4.1, pp. 217–218] has shown that the converse holds in case M is a compact manifold: a compact complex manifold is hyperbolic if and only if it contains no (nonconstant) entire holomorphic curves. Using Brody's technique, M. L. Green [3, Theorem 1, pp. 617–618] has proven a strong converse for subspaces of a torus: a closed complex subspace of a complex torus is hyperbolic if and only if it contains no (nontrivial) complex subtori. The following theorem provides a strong converse for convex domains in $\mathbf{C}^n$.

THEOREM. *Let M be a geometrically convex domain in $\mathbf{C}^n$ containing no complex affine lines. Then the Carathéodory pseudodistance c_M [4, p. 49] is a distance, and every closed ball with respect to c_M is compact. In particular, M is hyperbolic.*

PROOF. (a) The Carathéodory pseudodistance is a distance. Let L be the complex affine line joining two distinct points p and q of M. Since L is not contained in M, the complex affine line L contains a boundary point b of M. A real supporting hyperplane to M at b splits L into two (open complex) half lines, one of which (say H) contains $L \cap M$. This real supporting hyperplane contains a (unique) complex affine hyperplane P through b. The projection π of $\mathbf{C}^n$ onto L parallel to P maps M holomorphically into the half line H. Thus $c_M(p, q) \geqslant c_H(p, q) > 0$.

(b) Closed Carathéodory balls are compact. Given a sequence $\{q_k\}$ in M satisfying $c_M(p, q_k) \leqslant r$ for some point p of M and some number $r > 0$, we must extract a subsequence converging to a point of M. We may assume that $p = 0$ and that $q_k \neq p$ for all k. Let $\| \ \|$ denote the euclidean norm on $\mathbf{C}^n$. By taking a subsequence, we may assume that $v_k = q_k/\|q_k\| \to v$ with $\|v\| = 1$. Let L be the complex affine line joining $p = 0$ and v. Since L is not contained in M, the

Received by the editors September 22, 1979.
AMS (MOS) subject classifications (1970). Primary 32H15, 32H20; Secondary 32F15, 52A20.
Key words and phrases. Carathéodory pseudodistance, Kobayashi pseudodistance, convex domain, pseudoconvex domain.

© 1980 American Mathematical Society
0002-9939/80/0000-0360/$01.75

complex affine line L contains a boundary point b of M. Constructing the half line H and the projection π as in part (a), we obtain $r > c_M(p, q_k) > c_H(p, \pi(q_k))$. Because closed balls with respect to c_H are compact, we may assume that the sequence $\{\pi(q_k)\}$ converges to a point q of H. Noting that the mapping π is linear, we obtain $\pi(q_k) = \|q_k\|\pi(v_k)$. Since $\pi(q_k) \to q$ and $\pi(v_k) \to \pi(v) = v$ with $\|v\| = 1$, we see that $\|q_k\|$ is bounded. The expansion

$$q_k = \|q_k\|(v_k - \pi(v_k)) + \|q_k\|\pi(v_k)$$

now shows that the original sequence $\{q_k\}$ also converges to q. Finally we note that q is a point of M. For otherwise, q would lie on the boundary of M and we could have taken $b = q$ above; this is impossible because the boundary point b cannot belong to the (open complex) half line H. $\square$

The ideas involved in this proof can also be used to create a one-to-one holomorphic mapping of such a domain into a product of complex half lines. Thus every hyperbolic convex domain is biholomorphically equivalent to a bounded domain, and this theorem does not lead to any new examples of hyperbolic complex manifolds. It only makes them easier to construct and recognize.

This theorem and the positive results of Brody and Green tempt one to conjecture that every pseudoconvex domain containing no (nonconstant) entire holomorphic curves must be hyperbolic. The plausibility of this conjecture is supported by the Eisenman-Taylor [**4**, p. 130] and Campbell-Howard-Ochiai [**2**, p. 107] examples of nonhyperbolic manifolds containing no (nonconstant) entire holomorphic curves. Indeed, each of these examples is constructed by creating a nonhyperbolic manifold containing only a few curves and removing such small pieces of these curves that the nonhyperbolicity is undisturbed; removing the small pieces destroys local pseudoconvexity. Nevertheless, the following example disproves the conjecture.

Example. The formula

$$u(z) = \max\left(\log|z|, \sum_{1}^{\infty} k^{-2}\log|z - 1/k|\right)$$

defines a real-valued subharmonic function on the unit disk $D = \{z \mid |z| < 1\}$; hence the domain

$$M = \{(z, w) \in D \times \mathbf{C} \mid |w| < \exp(-u(z))\}$$

is pseudoconvex. Since $\exp(-u(1/k)) = \exp(-\log(1/k)) = k$, the mappings $f_k(\zeta) = (1/k, k\zeta)$ $(k = 1, 2, \dots)$ take D into M. For all points of the form $(0, w)$ in M we have

$$\begin{aligned} d_M((0,0),(0,w)) &= \lim d_M((1/k, 0), (1/k, w)) \\ &= \lim d_M(f_k(0), f_k(w/k)) \\ &< \lim d_D(0, w/k) = 0, \end{aligned}$$

proving that M is nonhyperbolic. On the other hand, if $f = (g, h)$: $\mathbf{C} \to M$ is an entire holomorphic curve, then g is bounded, hence constant (say $\equiv a$) by Liouville's theorem; it follows that

$$|h(z)| < \exp(-u(a)),$$

so that h is also bounded, hence constant. Thus M contains no nonconstant entire holomorphic curves.

References

1. R. Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219.

2. L. A. Campbell, A. Howard and T. Ochiai, *Moving holomorphic disks off analytic subsets*, Proc. Amer. Math. Soc. **60** (1976), 106–108.

3. M. L. Green, *Holomorphic maps to complex tori*, Amer. J. Math. **100** (1978), 615–620.

4. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Appl. Math., Vol. 2, Dekker, New York, 1970.

Department of Mathematics, University of California, Riverside, California 92521