APPROXIMATING MAPS AND A STONE-WEIERSTRASS THEOREM FOR C*-ALGEBRAS

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Abstract. Let $A$ be a $C^*$-algebra with identity and $B$ a $C^*$-subalgebra of $A$ which separates the pure states of $A$. We give an easy proof of the fact that, assuming there is a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$, $B$ must equal $A$. As corollaries we prove that if $B$ separates the pure states of $A$, then $B = A$ if $B$ is nuclear, or if $B = C^*_r(F_2)$ and $A \subseteq VN(F_2)$, where $F_2$ is the free group on two generators.

Let $A$ be a $C^*$-algebra with identity and let $B$ be a $C^*$-subalgebra of $A$ which contains the identity of $A$. Assume that $B$ separates the pure states of $A$. The Stone-Weierstrass problem is to show that $B$ must equal $A$. The main result of this paper (Theorem 3) is that $B$ must equal $A$ if, in addition, there is a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$. In the case that $A$ is separable, this result follows from a result of Effros [7, Theorem 11.1]. However, we offer a more elementary proof that we think is of interest. Our proof uses less specialized techniques and consists of first showing that $B$ separates the extreme points of the unit ball of $A^*$, then using a general functional analytic lemma of Wulbert [13], and finally an application of Rainwater's theorem [9, p. 33].

We give two corollaries of the main theorem. Let $B$ be a nuclear separable $C^*$-algebra which separates the pure states of $A$. The first corollary is that $B$ must equal $A$. This result was first proved, using reduction theory, by Sakai in [11]. Let $C^*_r(F_2)$ be the $C^*$-algebra generated by the left regular representation of the free group on two generators and let $VN(F_2)$ be the von Neumann algebra generated by $C^*_r(F_2)$. The second corollary is that if $C^*_r(F_2) \subseteq A \subseteq VN(F_2)$ and $C^*_r(F_2)$ separates the pure states of $A$, then $C^*_r(F_2)$ equals $A$. This situation is covered by the theorem because of a slight elaboration of a result of Haagerup [8].

The paper concludes with some partial results on a conjecture of Arveson concerning convergence of a completely positive approximation method for all compact operators when the method is known to converge for all operators in an irreducible set of compacts.

Throughout the paper $A^*$ will denote the Banach dual space of $A$, and $S(A)$ will denote the state space of $A$, i.e., the set of positive linear functionals on $A$ of norm

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one. We will use $e$ to denote the identity of a unital $C^*$-algebra. For $f$ in $A^*$ and $a$ in $A$, $f \cdot a$ is the element of $A^*$ defined by $(f \cdot a)(b) = f(ab)$ for all $b$ in $A$. Let $U(A)$ denote the set of unitaries in $A$. For $f$ in $A^*$, $|f|$ will denote the absolute value of $f$ [6, Definition 12.2.8]. For $X$ any Banach space, $X_1$ will denote the unit ball of $X$. For $S$ any convex set $S$ will denote the extreme points of $S$. Elements of $S(A)$ are called pure states of $A$. A set $B$ contained in $A$ is said to separate the pure states of $A$ if whenever $f, g \in S(A)$ and $f|B = g|B$, then $f = g$.

We first show why the main result follows, in the case that $A$ is separable, from [7, Theorem 11.1]. Let $B \subseteq A$, assume that $B$ separates the pure states of $A$ and that there exists a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$. Let $A_h$ denote the selfadjoint elements in $A$, and define $D: (B_h)^* \to (A_h)^*$ by $D(f)(a) = \text{LIM} f(L_n(a))$, where LIM is any generalized limit. Then $\|D(f)\| \leq \|f\|$, and $D(f)(b) = f(b)$ for all $b$ in $B$. If $f \in S(B)$, then $\|D(f)\| = 1$ and $D(f)(e) = 1$, so by [6, 2.1.9] $D(f) \in S(A)$. Hence $D$ is a dilation in the sense of [7, p. 20]. It is well known (see [6, Chapter 11]) that if $B$ separates the pure states of $A$, then the other hypotheses of [7, Theorem 11.1] are satisfied, so that $A = B$ if $A$ is separable.

The first step in our proof is the following lemma, which is obtained by combining [2, Theorem 2.1] and [10, Lemma 4 and its proof].

**Lemma 1.** Let $A$ be a $C^*$-algebra with identity. If $f \in \text{ext } A^*$ then $|f| \in \text{ext } S(A)$. If $B$ is a $C^*$-subalgebra of $A$ containing the identity such that $\pi_f$ restricted to $B$ is irreducible, then $f$ can be written in the form $|f| \cdot u$ for some unitary $u$ in $B$. Conversely, $f \cdot u \in \text{ext } A^*$ for any $f$ in $\text{ext } S(A)$ and $u$ in $U(A)$.

**Proof.** Let $f, g$ be in $\text{ext } A^*$ and assume $f|B = g|B$. By Lemma 1 and [6, 11.1.7 and 11.1.1], $f = |f| \cdot u$, $g = |g| \cdot v$ for $u$ and $v$ unitary elements of $B$. Then $f(u^*v) = g(u^*v)$ so $|f|(uv^*) = g(v^*) = |g|(e) = 1$. Since $uv^*$ is unitary it follows that $uv^* - e$ is in the left kernel of $|f|$ and $|f|(a) = |f|(uv^*a)$ for all $a$ in $A$. So for $b$ in $B$, we have

$$|g|(b) = |g|(uv^*b) = g(v^*b) = f(v^*b) = |f|(uv^*b) = |f|(b).$$

But by Lemma 1, $|f|$ and $|g|$ are in $\text{ext } S(A)$, so by hypothesis $|f| = |g|$. Thus for $a$ in $A$,

$$f(a) = |f|(ua) = |f|(uv^*va) = |f|(va) = g(va) = g(a),$$

so $f = g$ and we have proved that $B$ separates $\text{ext } A^*$. This fact and an elementary extreme point argument imply that each element of $\text{ext } B^*_1$ has a unique extension to an element of $A^*_1$. The last statement follows from Lemma 1 and [6, 11.1.7 and 11.1.1].
**Theorem 3.** Let $B$ be a unital $C^*$-subalgebra of a unital $C^*$-algebra $A$. Assume that $B$ separates the pure states of $A$. If there exists a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$, then $B = A$.

**Proof.** It follows from Lemma 2 and a general functional analysis argument of Wulbert [13, Lemma 1, part (i)] that $f(L_n(a))$ converges to $f(a)$ for each $f$ in $\text{ext } A^*$ and each $a$ in $A$. By Rainwater’s theorem, see [9, p. 33], this implies that $L_n(a)$ converges weakly to $a$ for each $a$ in $A$. But if $f \in A^*$ and $f|B = 0$, this then implies that $f(a) = \lim f(L_n(a)) = 0$, so $B$ must equal $A$.

In particular, if there is a norm one projection of $A$ onto $B$ and $B$ separates the pure states of $A$, then $B = A$, see [1, Theorem III.9]. The following corollary was first proved by Sakai in [11].

**Corollary 4.** Let $B$ be a nuclear separable $C^*$-algebra unitally contained in a $C^*$-algebra $A$. If $B$ separates the pure states of $A$, then $B = A$.

**Proof.** By [5] there is a sequence of finite-dimensional $C^*$-algebras $M_n$ and unital completely positive maps $S_n: B \to M_n$, $T_n: M_n \to B$ such that $T_n \circ S_n$ converges in the point-norm topology to the identity map on $B$. (This can be taken as the definition of nuclearity.) By [3, Theorem 1.2.3] there is a completely positive map $S': A \to M_n$ with $S' = S_n$ extending $S_n$. Let $L_n = T_n \circ S_n$. Then $L_n$ has norm one and $L_n(b)$ converges to $B$ in norm for each $b$ in $B$. Theorem 3 then implies that $B = A$.

For the second corollary of Theorem 3 we need to recall and elaborate slightly on some results of Haagerup [8]. We consider the left regular representation $\lambda$ of a countable discrete group $G$. Let $\delta_t \in l^2(G)$ be the function which is one at $t$ and zero elsewhere. For $s$ in $G$, $\lambda(s)$ is the unitary operator on $l^2(G)$ defined by $\lambda(s)\delta_t = \delta_{st}$. We denote by $C^*_\lambda(G)$ the $C^*$-algebra generated by the $\lambda(s)$, $s$ in $G$, and by $VN(G)$ the von Neumann algebra generated by $C^*_\lambda(G)$. Let $\phi$ be a positive definite function of $G$. Then it is shown in [8, Lemma 1.1] that there is a completely positive map $M_\phi: C^*_\lambda(G) \to C^*_\lambda(G)$ such that $M_\phi\lambda(s) = \phi(s)\lambda(s)$. The same proof shows that there is a unique ultra weakly continuous positive map $M_\phi: VN(G) \to VN(G)$ such that $M_\phi\lambda(s) = \phi(s)\lambda(s)$. For $\psi$ any finitely supported function on $G$ we can define $M_\psi: VN(G) \to C^*_\lambda(G)$ by $M_\psi(T) = \sum \psi(s)(T\delta_s)\lambda(s)$. Clearly $M_\psi$ is bounded and ultra weakly continuous.

Let $G$ be a countable discrete group. For $T$ in $VN(G)$, $T(f) = (T\delta_s) \circ f$ for all $f$ in $l^2(G)$. Conversely, if $g \in l^2(G)$ is such that $g$ convolves $l^2(G)$ into $l^2(G)$, then $g$ determines a bounded operator $c(g)$ in $VN(G)$ given by $c(g)(f) = g \circ f$. Hence $VN(G)$ can be identified with the set of functions in $l^2(G)$ which convolve $l^2(G)$ into $l^2(G)$, and $\|f\|_2 < \|c(f)\|_2$.

Let $F_2$ be the free group on two generators. For $s$ in $F_2$ let $|s|$ denote the length of (the reduced word for) $s$. If $f$ is a complex-valued function on $F_2$ with finite support, then [8, Lemma 1.5] states that

\[
\left\| \sum_{s \in F_2} f(s)\lambda(s) \right\| < 2 \left( \sum_{s \in G} |f(s)|^2(1 + |s|^4) \right)^{1/2}. \quad (*)
\]
For $\phi$ a positive definite function on $F_2$ let $\phi_n(s) = \phi(s)$ if $|s| < n$, $\phi_n(s) = 0$ if $|s| > n$. Then, by (\ast), for $f$ a function on $F_2$ with finite support we have (as in [8, Lemma 1.7]) that

$$\| M_{\phi_n}(c(f)) - M_\phi(c(f)) \| < 2K(\phi, n)\|f\|_2 < 2K(\phi, n)\|c(f)\|,$$

where $K(\phi, n) = \sup_{s \in F_2} |\phi_n(s) - \phi(s)| (1 + |s|)^2$.

Now let $c(f)$ be any element of $VN(F_2)$. Then by the Kaplansky density theorem there is a net $f_\alpha$ of finitely supported functions on $F_2$ such that $\|c(f_\alpha)\| < \|c(f)\|$ and $c(f_\alpha)$ converges to $c(f)$ in the strong operator topology. Then by the above

$$\| M_{\phi_n}(c(f_\alpha)) - M_\phi(c(f_\alpha)) \| < 2K(\phi, n)\|c(f)\|.$$ But since $M_{\phi_n}$ and $M_\phi$ are ultra-weakly continuous, it follows that $\| M_{\phi_n}(c(f)) - M_\phi(c(f)) \| < 2K(\phi, n)\|c(f)\|$. Now, as in [8, Theorem 1.8], let $\phi_\lambda(s) = e^{-\lambda|s|}$. Then $\phi_\lambda$ is a positive definite function on $F_2$ and

$$K(\phi_\lambda, n) = \sup_{|s| > n} e^{-\lambda|s|} (1 + |s|)^2,$$

so $K(\phi_\lambda, n)$ converges to zero as $n$ goes to infinity, for fixed $\lambda$. Hence $M_{\phi_\lambda}(c(f))$ is the norm limit of the truncated sums $M_{\phi_n}(c(f))$, so $M_{\phi_\lambda}(c(f))$ belongs to $C^*_\pi(F_2)$.

Lemma 5. There is a sequence of unital completely positive ultra-weakly continuous linear maps $L_n: VN(F_2) \to C^*_\pi(F_2)$ such that $L_n(b)$ converges to $b$ in norm for all $b$ in $C^*_\pi(F_2)$.

Corollary 6. If $A$ is a C*-algebra, $C^*_\pi(F_2) \subseteq A \subseteq VN(F_2)$ and $C^*_\pi(F_2)$ separates the pure states of $A$, then $C^*_\pi(F_2) = A$.

Let $H$ be a Hilbert space, $K(H)$ the compact operators on $H$, $B(H)$ the bounded operators on $H$. Let $S$ be an irreducible set of compact operators on $H$. Let $L_n: B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(s)$ converges to $s$ in norm for each $s$ in $S$. W. B. Arveson has asked if it follows that $L_n(a)$ converges to $a$ in norm for all compact operators $a$. We cannot answer this question, but we can prove the following two propositions along this line.

Proposition 7. Let $S$ be an irreducible set of compact operators acting on a Hilbert space $H$. Let $L_n: B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(s)$ converges to $s$ in the weak operator topology for all $s$ in $S$. Then $L_n(a)$ converges to $a$ in the weak operator topology for all compact operators $a$.

Proof. Let $m$ be any state on $l^\infty$ which is zero on $c_0$. Let $x$ and $y$ be in $H$ and define $L: B(H) \to B(H)$ by $(L(t)x, y) = m((L_n(t)x, y))$. Then $L$ is completely positive and $L(t) = t$ for all $t$ in $S \cup \{I\}$. But by [4, Remark 2, p. 288], the set of fixed points of $L$ is a C*-algebra. Since $S$ is irreducible this implies that $L(a) = a$ for all compact operators $a$. Hence $m((L_n(a)x, y)) = (ax, y)$ for all states $m$ on $l^\infty$ which are zero on $c_0$. It follows that $(L_n(a)x, y)$ converges to $(ax, y)$, so that $L_n(a)$ converges to $a$ in the weak operator topology for all compact $a$. 

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Proposition 8. Let $S$ be an irreducible set of compact operators and let $L_n : B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(K(H)) \subseteq S$ and $L_n(s)$ converges to $s$ in norm for each $s$ in $S$. Then $L_n(a)$ converges to $a$ in norm for each compact operator $a$.

Proof. By Proposition 9, $L_n(a)$ converges to $a$ in the weak operator topology for all compact operators $a$. We will use this to show that if $f_1$ and $f_2$ are two states on $K(H) + CI$ such that $f_1|S = f_2|S$, then $f_1 = f_2$. By [12, Theorem 3.4] this will imply that $L_n(a)$ converges in norm to $a$ for each compact operator $a$. So we assume that $f_1$ and $f_2$ are states on $K(H) + CI$ which are equal when restricted to $S$. Write $f_i = g_i + h_i$, where $g_i$ and $h_i$ are positive linear functionals with $g_i$ ultraweakly continuous and $h_i|K(H) = 0$. Then for $a$ in $K(H)$ we have

\[ g_1(a) = \lim (g_1(L_n(a))) = \lim (f_1(L_n(a))) = \lim (f_2(L_n(a))) = \lim (g_2(L_n(a))) = g_2(a). \]

So $g_1 = g_2$ and it follows that $f_1 = f_2$.

References

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