Let $A$ be a $C^*$-algebra with identity and $B$ a $C^*$-subalgebra of $A$ which separates the pure states of $A$. Assume that $B$ separates the pure states of $A$. The Stone-Weierstrass problem is to show that $B$ must be equal to $A$. The main result of this paper (Theorem 3) is that $B$ must equal $A$ if, in addition, there is a sequence of norm one linear maps $L_n: A \rightarrow B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$. In the case that $A$ is separable, this result follows from a result of Effros [7, Theorem 11.1]. However, we offer a more elementary proof that we think is of interest. Our proof uses less specialized techniques and consists of first showing that $B$ separates the extreme points of the unit ball of $A^*$, then using a general functional analytic lemma of Wulbert [13], and finally an application of Rainwater's theorem [9, p. 33].

We give two corollaries of the main theorem. Let $B$ be a nuclear separable $C^*$-algebra which separates the pure states of $A$. The first corollary is that $B$ must be equal to $A$. This result was first proved, using reduction theory, by Sakai in [11]. Let $C^*_r(F_2)$ be the $C^*$-algebra generated by the left regular representation of the free group on two generators and let $VN(F_2)$ be the von Neumann algebra generated by $C^*_r(F_2)$. The second corollary is that if $C^*_r(F_2) \subseteq A \subseteq VN(F_2)$ and $C^*_r(F_2)$ separates the pure states of $A$, then $C^*_r(F_2)$ equals $A$. This situation is covered by the theorem because of a slight elaboration of a result of Haagerup [8].

The paper concludes with some partial results on a conjecture of Arveson concerning convergence of a completely positive approximation method for all compact operators when the method is known to converge for all operators in an irreducible set of compacts.

Throughout the paper $A^*$ will denote the Banach dual space of $A$, and $S(A)$ will denote the state space of $A$, i.e., the set of positive linear functionals on $A$ of norm 1.
one. We will use $e$ to denote the identity of a unital $C^*$-algebra. For $f$ in $A^*$ and $a$ in $A$, $f \cdot a$ is the element of $A^*$ defined by $(f \cdot a)(b) = f(ab)$ for all $b$ in $A$. Let $U(A)$ denote the set of unitaries in $A$. For $f$ in $A^*$, $|f|$ will denote the absolute value of $f$ \cite[Definition 12.2.8]{6}. For $X$ any Banach space, $X_1$ will denote the unit ball of $X$.

For $S$ any convex set ext $S$ will denote the extreme points of $S$. Elements of ext $S(A)$ are called pure states of $A$. A set $B$ contained in $A$ is said to separate the pure states of $A$ if whenever $f, g \in$ ext $S(A)$ and $f|B = g|B$, then $f = g$.

We first show why the main result follows, in the case that $A$ is separable, from \cite[Theorem 11.1]{7}. Let $B \subseteq A$, assume that $B$ separates the pure states of $A$ and that there exists a sequence of norm one linear maps $L_n : A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$. Let $A_h$ denote the selfadjoint elements in $A$, and define $D : (B_h)^* \to (A_h)^*$ by $D(f)(a) = \lim L(f(a))$, where $\lim$ is any generalized limit. Then $\|D(f)\| \leq \|f\|$, and $D(f)(b) = f(b)$ for all $b$ in $B$. If $f \in S(B)$, then $\|D(f)\| = 1$ and $D(f)(e) = 1$, so by \cite[2.1.9]{6} $D(f) \in S(A)$. Hence $D$ is a dilation in the sense of \cite[p. 20]{7}. It is well known (see \cite[Chapter 11]{6}) that if $B$ separates the pure states of $A$, then the other hypotheses of \cite[Theorem 11.1]{7} are satisfied, so that $A = B$ if $A$ is separable.

The first step in our proof is the following lemma, which is obtained by combining \cite[Theorem 2.1]{2} and \cite[Lemma 4 and its proof]{10}.

**Lemma 1.** Let $A$ be a $C^*$-algebra with identity. If $f \in$ ext $A^*$ then $|f| \in$ ext $S(A)$. If $B$ is a $C^*$-subalgebra of $A$ containing the identity such that $\pi|_B$ restricted to $B$ is irreducible, then $f$ can be written in the form $|f| \cdot u$ for some unitary $u$ in $B$. Conversely, $f \cdot u \in$ ext $A^*$ for any $f$ in ext $S(A)$ and $u$ in $U(A)$.

**Lemma 2.** If $B$ separates the pure states of a $C^*$-algebra $A$ with identity, then $B$ separates ext $A^*$, each element of ext $B^*$ has a unique extension to an element of $A^*$, and each element of ext $A^*$ restricts to an element of ext $B^*$.

**Proof.** Let $f, g$ be in ext $A^*$ and assume $f|B = g|B$. By Lemma 1 and \cite[11.1.7 and 11.1.1]{6}, $f = |f| \cdot u$, $g = |g| \cdot v$ for $u$ and $v$ unitary elements of $B$. Then $f(v^*) = g(v^*)$ so $|f|(uv^*) = |g|(v^*) = |g|(e) = 1$. Since $uv^*$ is unitary it follows that $vu^* - e$ is in the left kernel of $|f|$ and $|f|(a) = |f|(uv^*a)$ for all $a$ in $A$. So for $b$ in $B$, we have

$$|g|(b) = |g|(uv^*b) = g(v^*b)$$

$$f(v^*b) = |f|(uv^*b) = |f|(b).$$

But by Lemma 1, $|f|$ and $|g|$ are in ext $S(A)$, so by hypothesis $|f| = |g|$. Thus for $a$ in $A$,

$$f(a) = |f|(ua) = |f|(uv^*a)$$

$$= |f|(va) = |g|(va) = g(a),$$

so $f = g$ and we have proved that $B$ separates ext $A^*$. This fact and an elementary extreme point argument imply that each element of ext $B^*$ has a unique extension to an element of $A^*$. The last statement follows from Lemma 1 and \cite[11.1.7 and 11.1.1]{6}.
**Theorem 3.** Let $B$ be a unital $C^*$-subalgebra of a unital $C^*$-algebra $A$. Assume that $B$ separates the pure states of $A$. If there exists a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to $b$ for each $b$ in $B$, then $B = A$.

**Proof.** It follows from Lemma 2 and a general functional analysis argument of Wulbert [13, Lemma 1, part (i)] that $f(L_n(a))$ converges to $f(a)$ for each $f$ in $\text{ext } A^*$ and each $a$ in $A$. By Rainwater’s theorem, see [9, p. 33], this implies that $L_n(a)$ converges weakly to $a$ for each $a$ in $A$. But if $f \in A^*$ and $f|B = 0$, this then implies that $f(a) = \lim f(L_n(a)) = 0$, so $B$ must equal $A$.

In particular, if there is a norm one projection of $A$ onto $B$ and $B$ separates the pure states of $A$, then $B = A$, see [1, Theorem III.9]. The following corollary was first proved by Sakai in [11].

**Corollary 4.** Let $B$ be a nuclear separable $C^*$-algebra unitally contained in a $C^*$-algebra $A$. If $B$ separates the pure states of $A$, then $B = A$.

**Proof.** By [5] there is a sequence of finite-dimensional $C^*$-algebras $M_n$ and unital completely positive maps $S_n: B \to M_n$, $T_n: M_n \to B$ such that $T_n \circ S_n$ converges in the point-norm topology to the identity map on $B$. (This can be taken as the definition of nuclearity.) By [3, Theorem 1.2.3] there is a completely positive map $S'_n: A \to M_n$ with $S'_n$ extending $S_n$. Let $L_n = T_n \circ S'_n$. Then $L_n$ has norm one and $L_n(b)$ converges to $B$ in norm for each $b$ in $B$. Theorem 3 then implies that $B = A$.

For the second corollary of Theorem 3 we need to recall and elaborate slightly on some results of Haagerup [8]. We consider the left regular representation $\lambda$ of a countable discrete group $G$. Let $\delta_t \in l^2(G)$ be the function which is one at $t$ and zero elsewhere. For $s$ in $G$, $\lambda(s)$ is the unitary operator on $l^2(G)$ defined by $\lambda(s)\delta_t = \delta_{st}$. We denote by $C^*_\rho(G)$ the $C^*$-algebra generated by the $\lambda(s)$, $s$ in $G$, and by $VN(G)$ the von Neumann algebra generated by $C^*_\rho(G)$. Let $\phi$ be a positive definite function of $G$. Then it is shown in [8, Lemma 1.1] that there is a completely positive map $\Phi: C^*_\rho(G) \to C^*_\lambda(G)$ such that $\Phi(\lambda(s)) = \phi(s)\lambda(s)$. The same proof shows that there is a unique ultraweakly continuous positive map $\Phi: VN(G) \to VN(G)$ such that $\Phi(\lambda(s)) = \phi(s)\lambda(s)$.

For any finitely supported function $\phi$ on $G$ we can define $\Phi: VN(G) \to C^*_\lambda(G)$ by $\Phi\phi(T) = \sum s \phi(s)\lambda(s)\lambda(s)$. Clearly, $\Phi\phi$ is bounded and ultraweakly continuous.

Let $G$ be a countable discrete group. For $T$ in $VN(G)$, $T(f) = (T\delta_t) \cdot f$ for all $f$ in $l^2(G)$. Conversely, if $g \in l^2(G)$ is such that $g$ convolves $l^2(G)$ into $l^2(G)$, then $g$ determines a bounded operator $c(g)$ in $VN(G)$ given by $c(g)(f) = g \cdot f$. Hence $VN(G)$ can be identified with the set of functions in $l^2(G)$ which convolve $l^2(G)$ into $l^2(G)$, and $\|f\| \leq \|c(f)\|$. For the results of Haagerup [8, Lemma 1.5] states that

$$\left\| \sum_{s \in F_2} f(s)\lambda(s) \right\| < 2 \left( \sum_{s \in G} |f(s)|^2 (1 + |s|^4)^{1/2} \right)^{1/2}.$$  

(•)
For \( \phi \) a positive definite function on \( F_2 \) let \( \phi_n(s) = \phi(s) \) if \( |s| < n \), \( \phi_n(s) = 0 \) if \( |s| > n \). Then, by (\( \ast \)), for \( f \) a function on \( F_2 \) with finite support we have (as in [8, Lemma 1.7]) that
\[
\| M_{\phi_n}(c(f)) - M_{\phi}(c(f)) \| \leq 2K(\phi, n)\| f \|_2 \leq 2K(\phi, n)\| c(f) \|
\]
where \( K(\phi, n) = \sup_{x \in F_2} |\phi_n(s) - \phi(s)(1 + |s|)^2| \).

Now let \( c(f) \) be any element of \( VN(F_2) \). Then by the Kaplansky density theorem there is a net \( f_n \) of finitely supported functions on \( F_2 \) such that \( \| c(f_n) \| \leq \| c(f) \| \) and \( c(f_n) \) converges to \( c(f) \) in the strong operator topology. Then by the above
\[
\| M_{\phi_n}(c(f_n)) - M_{\phi}(c(f_n)) \| \leq 2K(\phi, n)\| c(f) \|
\]
But since \( M_{\phi_n} \) and \( M_{\phi} \) are ultraweakly continuous, it follows that \( \| M_{\phi_n}(c(f)) - M_{\phi}(c(f)) \| \leq 2K(\phi, n)\| c(f) \| \). Now, as in [8, Theorem 1.8], let \( \phi(\lambda) = e^{-\lambda|\cdot|} \). Then \( \phi(\lambda) \) is a positive definite function on \( F_2 \) and
\[
K(\phi, n, \lambda) = \sup_{|s| > n} e^{-\lambda|s|}(1 + |s|)^2,
\]
so \( K(\phi_n, n) \) converges to zero as \( n \) goes to infinity, for fixed \( \lambda \). Hence \( M_{\phi_n}(c(f)) \) is the norm limit of the truncated sums \( M_{\phi_n}(c(f)) \), so \( M_{\phi_n}(c(f)) \) belongs to \( C_\ast^r(F_2) \). But if \( c(f) \) is in \( C_\ast^r(F_2) \), then it was shown in [8, Theorem 1.8] that \( M_{\phi_n}(c(f)) \) converges to \( c(f) \) in norm as \( \lambda \) goes to zero. To summarize, we then have the following lemma.

**Lemma 5.** There is a sequence of unital completely positive ultraweakly continuous linear maps \( L_n : VN(F_2) \to C_\ast^r(F_2) \) such that \( L_n(b) \) converges to \( b \) in norm for all \( b \) in \( C_\ast^r(F_2) \).

**Corollary 6.** If \( A \) is a \( C^\ast \)-algebra, \( C_\ast^r(F_2) \subseteq A \subseteq VN(F_2) \) and \( C_\ast^r(F_2) \) separates the pure states of \( A \), then \( C_\ast^r(F_2) = A \).

Let \( H \) be a Hilbert space, \( K(H) \) the compact operators on \( H \), \( B(H) \) the bounded operators on \( H \). Let \( S \) be an irreducible set of compact operators on \( H \). Let \( L_n : B(H) \to B(H) \) be a sequence of unital completely positive maps such that \( L_n(s) \) converges to \( s \) in norm for each \( s \) in \( S \). W. B. Arveson has asked if it follows that \( L_n(a) \) converges to \( a \) in norm for all compact operators \( a \). We cannot answer this question, but we can prove the following two propositions along this line.

**Proposition 7.** Let \( S \) be an irreducible set of compact operators acting on a Hilbert space \( H \). Let \( L_n : B(H) \to B(H) \) be a sequence of unital completely positive maps such that \( L_n(s) \) converges to \( s \) in the weak operator topology for all \( s \) in \( S \). Then \( L_n(a) \) converges to \( a \) in the weak operator topology for all compact operators \( a \).

**Proof.** Let \( m \) be any state on \( l^\infty \) which is zero on \( c_0 \). Let \( x \) and \( y \) be in \( H \) and define \( L : B(H) \to B(H) \) by \( (L(t)x, y) = m((L_n(t)x, y)) \). Then \( L \) is completely positive and \( L(t) = t \) for all \( t \) in \( S \cup \{I\} \). But by [4, Remark 2, p. 288], the set of fixed points of \( L \) is a \( C^\ast \)-algebra. Since \( S \) is irreducible this implies that \( L(a) = a \) for all compact operators \( a \). Hence \( m((L_n(a)x, y)) = (ax, y) \) for all states \( m \) on \( l^\infty \) which are zero on \( c_0 \). It follows that \( L_n(a,x, y) \) converges to \( (ax, y) \), so that \( L_n(a) \) converges to \( a \) in the weak operator topology for all compact \( a \).
Proposition 8. Let $S$ be an irreducible set of compact operators and let $L_n: B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(K(H)) \subseteq S$ and $L_n(s)$ converges to $s$ in norm for each $s$ in $S$. Then $L_n(a)$ converges to $a$ in norm for each compact operator $a$.

Proof. By Proposition 9, $L_n(a)$ converges to $a$ in the weak operator topology for all compact operators $a$. We will use this to show that if $f_1$ and $f_2$ are two states on $K(H) + CI$ such that $f_1|S = f_2|S$, then $f_1 = f_2$. By [12, Theorem 3.4] this will imply that $L_n(a)$ converges in norm to $a$ for each compact operator $a$. So we assume that $f_1$ and $f_2$ are states on $K(H) + CI$ which are equal when restricted to $S$. Write $f_i = g_i + h_i$, where $g_i$ and $h_i$ are positive linear functionals with $g_i$ ultraweakly continuous and $h_i|K(H) = 0$. Then for $a$ in $K(H)$ we have

$$g_1(a) = \lim (g_1(L_n(a))) = \lim (f_1(L_n(a))) = \lim (f_2(L_n(a))) = \lim (g_2(L_n(a))) = g_2(a).$$

So $g_1 = g_2$ and it follows that $f_1 = f_2$.

References

8. U. Haagerup, An example of a non nuclear C*-algebra which has the metric approximation property, Invent. Math. 50 (1979), 279–293.

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