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APPROXIMATING MAPS AND A STONE-WEIERSTRASS THEOREM FOR C^*-ALGEBRAS[1]

JOHN W. BUNCE

ABSTRACT. Let A be a C^*-algebra with identity and B a C^*-subalgebra of A which separates the pure states of A. We give an easy proof of the fact that, assuming there is a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to b for each b in B, B must equal A. As corollaries we prove that if B separates the pure states of A, then $B = A$ if B is nuclear, or if $B = C_r^*(F_2)$ and $A \subseteq VN(F_2)$, where F_2 is the free group on two generators.

Let A be a C^*-algebra with identity and let B be a C^*-subalgebra of A which contains the identity of A. Assume that B separates the pure states of A. The Stone-Weierstrass problem is to show that B must equal A. The main result of this paper (Theorem 3) is that B must equal A if, in addition, there is a sequence of norm one linear maps $L_n: A \to B$ such that $L_n(b)$ converges weakly to b for each b in B. In the case that A is separable, this result follows from a result of Effros [7, Theorem 11.1]. However, we offer a more elementary proof that we think is of interest. Our proof uses less specialized techniques and consists of first showing that B separates the extreme points of the unit ball of A^*, then using a general functional analytic lemma of Wulbert [13], and finally an application of Rainwater's theorem [9, p. 33].

We give two corollaries of the main theorem. Let B be a nuclear separable C^*-algebra which separates the pure states of A. The first corollary is that B must equal A. This result was first proved, using reduction theory, by Sakai in [11]. Let $C_r^*(F_2)$ be the C^*-algebra generated by the left regular representation of the free group on two generators and let $VN(F_2)$ be the von Neumann algebra generated by $C_r^*(F_2)$. The second corollary is that if $C_r^*(F_2) \subseteq A \subseteq VN(F_2)$ and $C_r^*(F_2)$ separates the pure states of A, then $C_r^*(F_2)$ equals A. This situation is covered by the theorem because of a slight elaboration of a result of Haagerup [8].

The paper concludes with some partial results on a conjecture of Arveson concerning convergence of a completely positive approximation method for all compact operators when the method is known to converge for all operators in an irreducible set of compacts.

Throughout the paper A^* will denote the Banach dual space of A, and $S(A)$ will denote the state space of A, i.e., the set of positive linear functionals on A of norm

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one. We will use e to denote the identity of a unital C^*-algebra. For f in A^* and a in A, $f \cdot a$ is the element of A^* defined by $(f \cdot a)(b) = f(ab)$ for all b in A. Let $U(A)$ denote the set of unitaries in A. For f in A^*, $|f|$ will denote the absolute value of f [6, Definition 12.2.8]. For X any Banach space, X_1 will denote the unit ball of X. For S any convex set ext S will denote the extreme points of S. Elements of ext $S(A)$ are called pure states of A. A set B contained in A is said to separate the pure states of A if whenever $f, g \in \text{ext } S(A)$ and $f|B = g|B$, then $f = g$.

We first show why the main result follows, in the case that A is separable, from [7, Theorem 11.1]. Let $B \subseteq A$, assume that B separates the pure states of A and that there exists a sequence of norm one linear maps L_n: $A \to B$ such that $L_n(b)$ converges weakly to b for each b in B. Let A_h denote the selfadjoint elements in A, and define D: $(B_h)^* \to (A_h)^*$ by $D(f)(a) = \text{LIM}\, f(L_n(a))$, where LIM is any generalized limit. Then $\|D(f)\| < \|f\|$, and $D(f)(b) = f(b)$ for all b in B. If $f \in S(B)$, then $\|D(f)\| = 1$ and $D(f)(e) = 1$, so by [6, 2.1.9] $D(f) \in S(A)$. Hence D is a dilation in the sense of [7, p. 20]. It is well known (see [6, Chapter 11]) that if B separates the pure states of A, then the other hypotheses of [7, Theorem 11.1] are satisfied, so that $A = B$ if A is separable.

The first step in our proof is the following lemma, which is obtained by combining [2, Theorem 2.1] and [10, Lemma 4 and its proof].

LEMMA 1. *Let A be a C^*-algebra with identity. If $f \in$ ext A_1^* then $|f| \in$ ext $S(A)$. If B is a C^*-subalgebra of A containing the identity such that $\pi_{|f|}$ restricted to B is irreducible, then f can be written in the form $|f| \cdot u$ for some unitary u in B. Conversely, $f \cdot u \in$ ext A_1^* for any f in* ext $S(A)$ *and u in $U(A)$.*

LEMMA 2. *If B separates the pure states of a C^*-algebra A with identity, then B separates* ext A_1^*, *each element of* ext B_1^* *has a unique extension to an element of A_1^*, and each element of* ext A_1^* *restricts to an element of* ext B_1^*.

PROOF. Let f, g be in ext A_1^* and assume $f|B = g|B$. By Lemma 1 and [6, 11.1.7 and 11.1.1], $f = |f| \cdot u$, $g = |g| \cdot v$ for u and v unitary elements of B. Then $f(v^*) = g(v^*)$ so $|f|(uv^*) = g(v^*) = |g|(e) = 1$. Since uv^* is unitary it follows that $vu^* - e$ is in the left kernel of $|f|$ and $|f|(a) = |f|(uv^*a)$ for all a in A. So for b in B, we have

$$\begin{aligned}|g|(b) &= |g|(vv^*b) = g(v^*b)\\ &= f(v^*b) = |f|(uv^*b) = |f|(b).\end{aligned}$$

But by Lemma 1, $|f|$ and $|g|$ are in ext $S(A)$, so by hypothesis $|f| = |g|$. Thus for a in A,

$$\begin{aligned}f(a) &= |f|(ua) = |f|(uv^*va)\\ &= |f|(va) = |g|(va) = g(a),\end{aligned}$$

so $f = g$ and we have proved that B separates ext A_1^*. This fact and an elementary extreme point argument imply that each element of ext B_1^* has a unique extension to an element of A_1^*. The last statement follows from Lemma 1 and [6, 11.1.7 and 11.1.1].

THEOREM 3. *Let B be a unital C^*-subalgebra of a unital C^*-algebra A. Assume that B separates the pure states of A. If there exists a sequence of norm one linear maps $L_n\colon A \to B$ such that $L_n(b)$ converges weakly to b for each b in B, then $B = A$.*

PROOF. It follows from Lemma 2 and a general functional analysis argument of Wulbert [13, Lemma 1, part (i)] that $f(L_n(a))$ converges to $f(a)$ for each f in ext A_1^* and each a in A. By Rainwater's theorem, see [9, p. 33], this implies that $L_n(a)$ converges weakly to a for each a in A. But if $f \in A^*$ and $f|B = 0$, this then implies that $f(a) = \lim f(L_n(a)) = 0$, so B must equal A.

In particular, if there is a norm one projection of A onto B and B separates the pure states of A, then $B = A$, see [1, Theorem III.9]. The following corollary was first proved by Sakai in [11].

COROLLARY 4. *Let B be a nuclear separable C^*-algebra unitally contained in a C^*-algebra A. If B separates the pure states of A, then $B = A$.*

PROOF. By [5] there is a sequence of finite-dimensional C^*-algebras M_n and unital completely positive maps $S_n\colon B \to M_n$, $T_n\colon M_n \to B$ such that $T_n \circ S_n$ converges in the point-norm topology to the identity map on B. (This can be taken as the definition of nuclearity.) By [3, Theorem 1.2.3] there is a completely positive map $S_n'\colon A \to M_n$ with S_n' extending S_n. Let $L_n = T_n \circ S_n'$. Then L_n has norm one and $L_n(b)$ converges to B in norm for each b in B. Theorem 3 then implies that $B = A$.

For the second corollary of Theorem 3 we need to recall and elaborate slightly on some results of Haagerup [8]. We consider the left regular representation λ of a countable discrete group G. Let $\delta_t \in l^2(G)$ be the function which is one at t and zero elsewhere. For s in G, $\lambda(s)$ is the unitary operator on $l^2(G)$ defined by $\lambda(s)\delta_t = \delta_{st}$. We denote by $C_r^*(G)$ the C^*-algebra generated by the $\lambda(s)$, s in G, and by $VN(G)$ the von Neumann algebra generated by $C_r^*(G)$. Let ϕ be a positive definite function of G. Then it is shown in [8, Lemma 1.1] that there is a completely positive map $M_\phi\colon C_r^*(G) \to C_r^*(G)$ such that $M_\phi\lambda(s) = \phi(s)\lambda(s)$. The same proof shows that there is a unique ultraweakly continuous completely positive map M_ϕ: $VN(G) \to VN(G)$ such that $M_\phi\lambda(s) = \phi(s)\lambda(s)$. For ψ any finitely supported function on G we can define $M_\psi\colon VN(G) \to C_r^*(G)$ by $M_\psi(T) = \sum \psi(s)(T\delta_e)(s)\lambda(s)$. Clearly M_ψ is bounded and ultraweakly continuous.

Let G be a countable discrete group. For T in $VN(G)$, $T(f) = (T\delta_e) * f$ for all f in $l^2(G)$. Conversely, if $g \in l^2(G)$ is such that g convolves $l^2(G)$ into $l^2(G)$, then g determines a bounded operator $c(g)$ in $VN(G)$ given by $c(g)(f) = g * f$. Hence $VN(G)$ can be identified with the set of functions in $l^2(G)$ which convolve $l^2(G)$ into $l^2(G)$, and $\|f\|_2 < \|c(f)\|$.

Let F_2 be the free group on two generators. For s in F_2 let $|s|$ denote the length of (the reduced word for) s. If f is a complex-valued function on F_2 with finite support, then [8, Lemma 1.5] states that

$$\Big\| \sum_{s \in F_2} f(s)\lambda(s) \Big\| < 2\Big(\sum_{s \in G} |f(s)|^2 (1 + |s|)^4 \Big)^{1/2}. \qquad (*)$$

For ϕ a positive definite function on F_2 let $\phi_n(s) = \phi(s)$ if $|s| < n$, $\phi_n(s) = 0$ if $|s| > n$. Then, by (*), for f a function on F_2 with finite support we have (as in [8, Lemma 1.7]) that

$$\|M_{\phi_n}(c(f)) - M_{\phi}(c(f))\| < 2K(\phi, n)\|f\|_2 < 2K(\phi, n)\|c(f)\|,$$

where $K(\phi, n) = \sup_{s \in F_2}|\phi_n(s) - \phi(s)|(1 + |s|)^2$.

Now let $c(f)$ be any element of $VN(F_2)$. Then by the Kaplansky density theorem there is a net f_α of finitely supported functions on F_2 such that $\|c(f_\alpha)\| < \|c(f)\|$ and $c(f_\alpha)$ converges to $c(f)$ in the strong operator topology. Then by the above $\|M_{\phi_n}(c(f_\alpha)) - M_{\phi}(c(f_\alpha))\| < 2K(\phi, n)\|c(f)\|$. But since M_{ϕ_n} and M_ϕ are ultraweakly continuous, it follows that $\|M_{\phi_n}(c(f)) - M_{\phi}(c(f))\| < 2K(\phi, n)\|c(f)\|$. Now, as in [8, Theorem 1.8], let $\phi_\lambda(s) = e^{-\lambda|s|}$. Then ϕ_λ is a positive definite function on F_2 and

$$K(\phi_\lambda, n) = \sup_{|s|>n} e^{-\lambda|s|}(1 + |s|)^2,$$

so $K(\phi_\lambda, n)$ converges to zero as n goes to infinity, for fixed λ. Hence $M_{\phi_\lambda}(c(f))$ is the norm limit of the truncated sums $M_{\phi_{\lambda_n}}(c(f))$, so $M_{\phi_\lambda}(c(f))$ belongs to $C_r^*(F_2)$. But if $c(f)$ is in $C_r^*(F_2)$, then it was shown in [8, Theorem 1.8] that $M_{\phi_\lambda}(c(f))$ converges to $c(f)$ in norm as λ goes to zero. To summarize, we then have the following lemma.

LEMMA 5. *There is a sequence of unital completely positive ultraweakly continuous linear maps L_n: $VN(F_2) \to C_r^*(F_2)$ such that $L_n(b)$ converges to b in norm for all b in $C_r^*(F_2)$.*

COROLLARY 6. *If A is a C^*-algebra, $C_r^*(F_2) \subseteq A \subseteq VN(F_2)$ and $C_r^*(F_2)$ separates the pure states of A, then $C_r^*(F_2) = A$.*

Let H be a Hilbert space, $K(H)$ the compact operators on H, $B(H)$ the bounded operators on H. Let S be an irreducible set of compact operators on H. Let L_n: $B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(s)$ converges to s in norm for each s in S. W. B. Arveson has asked if it follows that $L_n(a)$ converges to a in norm for all compact operators a. We cannot answer this question, but we can prove the following two propositions along this line.

PROPOSITION 7. *Let S be an irreducible set of compact operators acting on a Hilbert space H. Let L_n: $B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(s)$ converges to s in the weak operator topology for all s in S. Then $L_n(a)$ converges to a in the weak operator topology for all compact operators a.*

PROOF. Let m be any state on l^∞ which is zero on c_0. Let x and y be in H and define L: $B(H) \to B(H)$ by $(L(t)x, y) = m((L_n(t)x, y))$. Then L is completely positive and $L(t) = t$ for all t in $S \cup \{I\}$. But by [4, Remark 2, p. 288], the set of fixed points of L is a C^*-algebra. Since S is irreducible this implies that $L(a) = a$ for all compact operators a. Hence $m((L_n(a)x, y)) = (ax, y)$ for all states m on l^∞ which are zero on c_0. It follows that $(L_n(a)x, y)$ converges to (ax, y), so that $L_n(a)$ converges to a in the weak operator topology for all compact a.

PROPOSITION 8. *Let S be an irreducible set of compact operators and let L_n: $B(H) \to B(H)$ be a sequence of unital completely positive maps such that $L_n(K(H)) \subseteq S$ and $L_n(s)$ converges to s in norm for each s in S. Then $L_n(a)$ converges to a in norm for each compact operator a.*

PROOF. By Proposition 9, $L_n(a)$ converges to a in the weak operator topology for all compact operators a. We will use this to show that if f_1 and f_2 are two states on $K(H) + \mathbf{C}I$ such that $f_1|S = f_2|S$, then $f_1 = f_2$. By [12, Theorem 3.4] this will imply that $L_n(a)$ converges in norm to a for each compact operator a. So we assume that f_1 and f_2 are states on $K(H) + \mathbf{C}I$ which are equal when restricted to S. Write $f_i = g_i + h_i$, where g_i and h_i are positive linear functionals with g_i ultraweakly continuous and $h_i|K(H) = 0$. Then for a in $K(H)$ we have

$$g_1(a) = \lim(g_1(L_n(a))) = \lim(f_1(L_n(a))) = \lim(f_2(L_n(a)))$$
$$= \lim(g_2(L_n(a))) = g_2(a).$$

So $g_1 = g_2$ and it follows that $f_1 = f_2$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045