ON DOMINATED EXTENSIONS IN FUNCTION ALGEBRAS

J. GLOBEVIK

ABSTRACT. The Bishop-Gamelin interpolation theorem asserts that given a compact Hausdorff space \( K \), a closed subspace \( A \) of \( C(K) \), a positive continuous function \( p \) on \( K \) and a closed set \( F \subset K \) such that every measure in the annihilator of \( A \) vanishes on \( F \), every function \( f \in C(F) \) satisfying \( |f(s)| < p(s) \) \((s \in F)\) extends to a function \( \tilde{f} \in A \) satisfying \( |	ilde{f}(z)| < p(z) \) \((z \in K)\). In the paper we consider a special case where the theorem is extended to the situation when the dominating function is nonnegative.

Let \( K \) be a compact Hausdorff space and \( A \) a closed subspace of \( C(K) \), the space of all (real- or complex-valued) continuous functions on \( K \), with sup norm. Let \( A^\perp \) denote the set of all annihilating measures of \( A \) and \( \mu_F \) the restriction of the measure \( \mu \) to \( F \), a closed subset of \( K \). If \( \mu \in A^\perp \) implies that \( \mu_F = 0 \) then

given any positive continuous function \( q \) on \( K \) and any \( f \in C(F) \) satisfying \( |f| < q \) on \( F \) there is some \( \tilde{f} \in A \) which extends \( f \) and satisfies \( |	ilde{f}| < q \) on \( K \).

This is a strengthened form of a well-known theorem of E. Bishop [2] which is a special case of a more general theorem proved by T. W. Gamelin [3]. Note that (1) implies (1) with \( < \) replaced by \( < \).

In known generalizations and applications of Bishop's theorem [1], [3], [4], [5], [6], [7], [9], [10], [11] the dominating functions \( q \) are always assumed to be positive on \( K \). Our purpose here is to present a special case where the conclusion of Bishop's theorem can be strengthened to allow domination by nonnegative functions.

Let \( K, F \) and \( A \) be as above and let \( p \) be a nonnegative continuous function on \( K \). Is it true that

given any \( f \in C(F) \) which satisfies \( |f| < p \) on \( F \) there exists an \( \tilde{f} \in A \) which extends \( f \) and satisfies \( |	ilde{f}| < p \) on \( K \)?

We begin with two examples. Let \( T \) be the unit circle in \( C \) and let \( B \subset C(T) \) be the disc algebra. By the F. and M. Riesz theorem any closed set \( F \subset T \) of Lebesgue measure 0 satisfies the condition \( \mu \in B^\perp \Rightarrow \mu_F = 0 \) [8]. Assume that \( p \) is a nonnegative continuous function on \( T \) which vanishes on a set of positive Lebesgue measure on \( T \) but does not vanish identically. Then any \( g \in B \) satisfying \( |g| < p \) on \( T \) vanishes on a set of positive Lebesgue measure and consequently...
vanishes identically [8]. This shows that (2) is false in general. If (2) is satisfied then clearly

for each \( x \in F - p^{-1}(0) \) there is some \( g \in A \) which satisfies
\[
g(x) \neq 0 \text{ and } |g| < p \text{ on } K.\tag{3}
\]

Suppose that (3) is satisfied. Let \( A \subset C(T) \) be the subspace spanned by \( B \) and a function \( h \in C(T) - B \) which vanishes on a set of positive Lebesgue measure but does not vanish identically. Again, since \( B \subset A \), any closed set \( F \subset T \) of Lebesgue measure 0 satisfies the condition \( \mu \in A^+ \Rightarrow \mu_F = 0 \). Put \( p(z) = |h(z)| \) (\( z \in T \)). Then \( A \) and \( p \) satisfy (3). Suppose that \( g \in A \) satisfies \( |g| < p \) on \( T \). Then \( g \) vanishes on the zero set of \( h \) and consequently \( g \) is a scalar multiple of \( h \). This shows that (3) is not a sufficient condition for (2) if \( A \) is a subspace of \( C(K) \). However, if \( A \) is a subalgebra of \( C(K) \) then (3) implies (2) and this is the main result of the present paper.

If \( A \) and \( F \) satisfy (1) then we will say that \( A \) has the dominated extension property with respect to \( F \) [10].

**Theorem.** Let \( K \) be a compact Hausdorff space and let \( A \subset C(K) \) be a closed subalgebra. Suppose that \( F \subset K \) is a closed set such that \( A \) has the dominated extension property with respect to \( F \). Let \( p \) be a nonnegative continuous function on \( K \) such that for any \( x \in F - p^{-1}(0) \) there is some \( v \in A \) satisfying \( v(x) \neq 0 \) and \( |v(z)| < p(z) \) (\( z \in K \)). Then given any \( f \in C(F) \) satisfying \( |f(s)| < p(s) \) (\( s \in F \)) there is an extension \( \tilde{f} \in A \) of \( f \) such that \( |	ilde{f}(z)| < p(z) \) (\( z \in K \)).

**Lemma.** Let \( K \) be a compact Hausdorff space, \( A \subset C(K) \) a closed subalgebra and \( G \subset K \) a closed set such that \( A \) has the dominated extension property with respect to \( G \). Let \( p \) be a nonnegative continuous function on \( K \) such that given any \( x \in G \) there is some \( v \in A \) satisfying \( v(x) \neq 0 \) and \( |v(z)| < p(z) \) (\( z \in K \)). Then given any neighbourhood \( U \) of \( G \) there is a function \( h \in A \) such that
\[
\begin{align*}
(i) \quad & ||h|| = 1, \\
(ii) \quad & h(s) = 1 \quad (s \in G), \\
(iii) \quad & |h(z)| < p(z) \quad (z \in K - U).
\end{align*}
\]

**Proof.** Assume for a moment that some \( g \in A \) satisfies \( g(z) \neq 0 \) (\( z \in G \)) and \( |g(z)| < p(z) \) (\( z \in K \)). Let \( U \) be a neighbourhood of \( G \). Passing to a smaller \( U \) if necessary we may assume with no loss of generality that \( U \) is open and that \( 1/|g| \) is bounded on \( U \). By the Urysohn lemma there is some \( \Phi \in C(K) \) such that \( 0 < \Phi < 1, \Phi|G = 1, \text{ and } \Phi|(K - U) = 0 \). Then
\[
z \to \phi(z) = \begin{cases} \\
\Phi(z)/|g(z)| & (z \in U), \\
0 & (z \in K - U),
\end{cases}
\]
is a nonnegative continuous function on \( K \). Choose \( \eta: 0 < \eta < 1 \) so that \( \eta \cdot ||g|| < 1 \) and let \( \psi(z) = \max(\eta, \phi(z)) \) (\( z \in K \)). The function \( \psi \) is positive and continuous on \( K \) and satisfies \( \psi(z) < 1/|g(z)| \) (\( z \in U \)). If \( z \in G \) then \( \psi(z) = \max(\eta, \phi(z)) = \max(\eta, 1/|g(z)|) = 1/|g(z)| \) so \( |1/g(z)| < \psi(z) \) (\( z \in G \)). Since \( A \) has the dominated extension property with respect to \( G \) there is some \( u \in A \) satisfying
u(z) = 1/g(z) (z ∈ G) and |u(z)| < ψ(z) (z ∈ K). Define h = u · g. Then h belongs to A and satisfies (ii). If z ∈ U then |h(z)| < ψ(z) · |g(z)| < (1/|g(z)|) · |g(z)| = 1. Let z ∈ K - U. Then ψ(z) = 0 so ψ(z) = η. Consequently |h(z)| < ψ(z) · |g(z)| = η · |g(z)|. Since η < 1 it follows that |h(z)| < |g(z)| < p(z) which proves (iii). Since η · ||g|| < 1 we also have |h(z)| < 1 which completes the proof of (i).

It remains to show that there is some g ∈ G such that g(z) ≠ 0 (z ∈ G) and |g(z)| < p(z) (z ∈ K). Clearly there is a neighbourhood V of G such that p(z) > ε (z ∈ V) for some ε > 0. By the assumptions and by the compactness of G there are v₁, v₂, . . . , vₙ ∈ A, vᵢ(z) < p(z) (z ∈ K), and positive numbers η₁, η₂, . . . , ηₙ such that Uᵢ = {x ∈ G: |vᵢ(x)| > ηᵢ} (1 < i < n) are nonempty (relatively) open subsets of G which cover G. Let {Φᵢ} be a subordinated partition of unity on G. By the properties of G each Φᵢ has an extension Φᵢ ∈ A. Define the closed sets Gᵢ ⊂ G by Gᵢ = {x ∈ G: |vᵢ(x)| > ηᵢ} (1 < i < n). Since A has the dominated extension property with respect to G it has the dominated extension property with respect to every closed subset of G, in particular, with respect to Gᵢ (1 < i < n). By the first part of the proof there are functions hᵢ ∈ A (1 < i < n) satisfying ∥hᵢ∥ = 1, hᵢ|Gᵢ = 1 and ∥Φᵢ∥ · |hᵢ(z)| < p(z)/n (z ∈ K - V). Since Uᵢ ⊂ Gᵢ (1 < i < n) it follows that hᵢ(z)Φᵢ(z) = Φᵢ(z) (z ∈ G, 1 < i < n). Define u = Σᵢ=₁ Φᵢ · hᵢ. Clearly u|G = 1. If z ∈ K - V then |u(z)| < Σᵢ=₁ |Φᵢ(z)| · |hᵢ(z)| < n · (p(z)/n) = p(z). It follows that g = min{1, ε/∥u∥} · u has all the required properties. Q.E.D.

Proof of Theorem. Denote by N the set of all nonnegative integers. The case f = 0 is trivial so with no loss of generality assume that supₙ∈F |f(z)| = 1 and |f(s)| < p(s) (s ∈ F). Then Fₙ = {z ∈ F: |f(z)| > 1/2ⁿ} is an increasing sequence of nonempty closed subsets of F disjoint from p⁻¹(0). By the Urysohn lemma there exist Φₙ ∈ C(F) (n ∈ N) such that for each n ∈ N

Φₙ(z) = 1 - 1/2ⁿ (z ∈ Fₙ),
0 < Φₙ(z) < 1 - 1/2ⁿ (z ∈ F),
Φₙ(z) = 0 (z ∈ F - Fₙ₊₁).

Note that the sequence Φₙ is increasing. Put fₙ(z) = Φₙ(z)f(z) (z ∈ F, n ∈ N). Clearly fₙ ∈ C(F) (n ∈ N) and by the properties of Φₙ we have

$$\lim_{n \to \infty} fₙ(s) = f(s) \quad (s \in F).$$

(4)

Let n ∈ N. If z ∈ Fₙ then

$$|fₙ₊₁(z) - fₙ(z)| = |Φₙ₊₁(z) - Φₙ(z)| \cdot |f(z)| < 1/2ⁿ⁺¹.$$

If z ∈ F - Fₙ then |f(z)| < 1/2ⁿ so

$$|fₙ₊₁(z) - fₙ(z)| = |Φₙ₊₁(z) - Φₙ(z)| \cdot |f(z)| < 1/2ⁿ.$$

It follows that

$$|fₙ₊₁(s) - fₙ(s)| < 1/2ⁿ \quad (s \in F, n \in N).$$

(5)
Suppose that we have constructed a sequence \( \tilde{f}_i \in A \ (i \in \mathbb{N}) \) such that for each \( i \in \mathbb{N} \)
(a) \( \tilde{f}_i[F = f_i \),
(b) \( \|\tilde{f}_{i+1} - \tilde{f}_i\| < 1/2^i \),
(c) \( |\tilde{f}_i(z)| < (1 - 1/2^{i+1})p(z) \ (z \in K - p^{-1}(0)) \),
(d) \( \tilde{f}_i(z) = 0 \ (z \in p^{-1}(0)) \).

Define \( \tilde{f}(z) = \lim \tilde{f}_i(z) \ (z \in K) \). Since \( \tilde{f}_i \in A \ (i \in \mathbb{N}) \) and since \( A \) is closed it follows by (b) that \( \tilde{f} \in A \). By (a) and (d) \( \tilde{f} \) extends \( f \). By (c) and (d) we have
\[
|\tilde{f}(z)| < p(z) \ (z \in K).
\]

It remains to prove the existence of \( \tilde{f} \) satisfying (a) through (d) above. Put \( \tilde{f}_0 = 0 \).

Clearly \( \tilde{f}_0 \) satisfies (a), (c) and (d) for \( i = 0 \). Let \( n \in \mathbb{N} \) and suppose that we have constructed \( \tilde{f}_n \in A \) satisfying (a), (c) and (d) for \( i = n \). By (c) we have
\[
(1 - 1/2^{n+2})p(z) - |\tilde{f}_n(z)| > 0 \quad (z \in K - p^{-1}(0)).
\]

Since the term on the left is continuous in \( z \) and since \( F_{n+2} \) is a compact set disjoint from \( p^{-1}(0) \) it follows that there is some \( \eta > 0 \) and a neighborhood \( U \) of \( F_{n+2} \) such that
\[
(1 - 1/2^{n+2})p(z) - |\tilde{f}_n(z)| > \eta \quad (z \in U). \tag{6}
\]

Define
\[
\phi(z) = \max\left\{ (1 - 1/2^{n+2})p(z) - |\tilde{f}_n(z)|, \eta \right\} \quad (z \in K).
\]

Clearly \( \phi \) is a positive continuous function on \( K \). Let \( s \in F_{n+2} \). Since \( p(s) > 0 \) it follows that
\[
|f_n(s)| + |f_{n+1}(s) - f_n(s)| = \Phi_n(s) \cdot |f(s)| + \left[ \Phi_{n+1}(s) - \Phi_n(s) \right] \cdot |f(s)|
\]
\[
= \Phi_{n+1}(s) \cdot |f(s)| < (1 - 1/2^{n+1})p(s)
\]
and consequently \( |f_{n+1}(s) - f_n(s)| < \phi(s) \ (s \in F_{n+2}) \). If \( s \in F - F_{n+2} \) then \( f_n(s) = f_{n+1}(s) = 0 \) so \( |f_{n+1}(s) - f_n(s)| < \phi(s) \ (s \in F - F_{n+2}) \). Define \( \psi(z) = \min\{1/2^n, \phi(z)\} \) \( (z \in K) \). Clearly \( \psi \) is a positive continuous function on \( K \). By (5) and by the preceding discussion it follows that \( |f_{n+1}(s) - f_n(s)| < \psi(s) \ (s \in F) \). By the dominated extension property of \( A \) with respect to \( F \) there is some \( u \in A \) such that
\[
u(s) = f_{n+1}(s) - f_n(s) \quad (s \in F), \tag{7}
\]
and
\[
|\nu(z)| < \psi(z) \quad (z \in K), \tag{8}
\]
which gives
\[
|\nu(z)| < \phi(z) \quad (z \in K), \tag{8}
\]
and
\[
\|\nu\| < 1/2^n. \tag{9}
\]

If \( u = 0 \) then put \( \tilde{f}_{n+1} = \tilde{f}_n \) and we are done. Assume that \( u \neq 0 \). Since \( A \) has the dominated extension property with respect to \( F \) it has the dominated extension
property with respect to $F_{n+2}$. By lemma it follows that there is some $h \in A$ which satisfies

\[ \|h\| = 1, \quad (10) \]

\[ h|_{F_{n+2}} = 1, \quad (11) \]

\[ |h(z)| \leq p(z) / (2^{n+2} \cdot \|u\|) \quad (z \in K - U). \quad (12) \]

Define $\tilde{f}_{n+1} = \tilde{f}_n + h \cdot u$. Clearly $\tilde{f}_{n+1} \in A$. By (7) and (11) we have $(hu)|_{F_{n+2}} = (\tilde{f}_{n+1} - f_n)|_{F_{n+2}}$. Since $(f_{n+1} - f_n)(F - F_{n+2}) = 0$ and since $\tilde{f}_n|F = f_n$ it follows that (a) is satisfied for $i = n + 1$. By (9) and (10) (b) also is satisfied for $i = n$. Observe that $p(z) > 0 \ (z \in U)$ so, by (12), $h(z) = 0 \ (z \in p^{-1}(0))$. Since $\tilde{f}_n(z) = 0 \ (z \in p^{-1}(0))$ it follows that (d) is satisfied for $i = n + 1$. Let $z \in U$. By (6) we have $\phi(z) = (1 - 1/2^{n+2})p(z) - |\tilde{f}_n(z)|$ which by (8) implies that $|u(z)| < (1 - 1/2^{n+2})p(z) - |\tilde{f}_n(z)|$. By (10) it follows that $|\tilde{f}_{n+1}(z)| < |\tilde{f}_n(z)| + |u(z)| < (1 - 1/2^{n+2})p(z)$ which proves (c) for $z \in U$ and $i = n + 1$. Finally, let $z \in (K - p^{-1}(0)) - U$. By (12) and by (c) for $i = n$ it follows that

\[ |\tilde{f}_{n+1}(z)| < |\tilde{f}_n(z)| + |h(z)| \cdot |u(z)| \]

\[ \leq (1 - 1/2^{n+1})p(z) + |u(z)| \cdot p(z) / (2^{n+2} \cdot \|u\|) \]

\[ \leq (1 - 1/2^{n+2})p(z) \]

which completes the proof of (c) for $i = n + 1$. Q.E.D.

**Remark 1.** The theorem still holds if we assume that $p$ is lower semicontinuous. Namely, (1) holds for all positive continuous functions $q$ iff it holds for all positive lower semicontinuous functions [10] so our proofs work also in the case when $p$ is lower semicontinuous.

**Remark 2.** By changing slightly the proofs of the lemma and theorem one can prove the following: Let $K, A$ and $F$ be as in the theorem. Suppose that $I \subset A$ is a closed ideal and let $p$ be a nonnegative continuous function on $K$ such that given any $x \in F - p^{-1}(0)$ there is some $v \in I$ which satisfies $v(x) \neq 0$ and $|v(z)| \leq p(z) \ (z \in K)$. Then given any $f \in C(F)$ satisfying $|f(z)| \leq p(z) \ (s \in F)$ there is an extension $\tilde{f} \in I$ of $f$ such that $|\tilde{f}(z)| \leq p(z) \ (z \in K)$. This sharpens [11, Theorem 20.12].

**Corollary.** Let $A$ be the disc algebra on the unit circle $T$ and let $p$ be a nonnegative continuous function on $T$ which does not vanish identically. The following are equivalent:

(i) The function $t \to \log p(t)$ is (Lebesgue) integrable on $T$.

(ii) Given any closed set $F \subset T$ of (Lebesgue) measure 0 and any continuous function $f \in C(F)$ satisfying $|f(s)| \leq p(s) \ (s \in F)$ there exists $\tilde{f} \in A$ which extends $f$ and satisfies $|\tilde{f}(z)| \leq p(z) \ (z \in K)$.

**Proof.** (ii) $\Rightarrow$ (i) follows from the fact that for any nonzero $g \in A$ the function $t \to \log |g(t)|$ is integrable [8]. To prove (i) $\Rightarrow$ (ii) let $p$ be a nonnegative continuous function on $T$ with log $p$ integrable. By the F. and M. Riesz theorem [8] and by the Bishop theorem $A$ has the dominated extension property with respect to every closed set $F \subset T$ of measure 0. So by our theorem it suffices to prove that given
any \( s \in T - p^{-1}(0) \) there is some \( g \in A \) such that \( g(s) \neq 0 \) and \( |g(z)| < p(z) \) \((z \in T)\). The following construction was suggested by D. Marshall. Writing \( T - p^{-1}(0) \) as a union of (at most countably many) pairwise disjoint open arcs and approximating \( \log|p| \) on each of these arcs by a suitable continuously differentiable function one can get a nonnegative continuous function \( q \) on \( T \) such that \( q < p \) on \( T \), \( \log q \) is integrable, \( q^{-1}(0) = p^{-1}(0) \) and \( q \) is continuously differentiable on \( T - p^{-1}(0) \). Now the boundary function \( g \) of the function

\[
z \to \exp\left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log q(e^{i\theta}) \, d\theta \right) \quad (|z| < 1)
\]

belongs to \( A \) and satisfies \( |g| < p \) on \( T \) and \( g^{-1}(0) = p^{-1}(0) \) [8]. Q.E.D.

The referee kindly suggested that the following two open problems could be mentioned.

1. In the case when \( K = T \) and \( A \) is the disc algebra on \( T \) the condition (3) in the theorem can be replaced by the following weaker condition: There exists \( g \in A \), \( g \neq 0 \), such that \( |g(z)| < p(z) \) \((z \in T)\). (This implies that \( t \to \log p(t) \) is Lebesgue integrable on \( T \) and the conclusion follows from the corollary.) Does this hold also in the case when \( K \) is the closed unit disc and \( A \) is the disc algebra on \( K \)? Simple examples show that this does not hold in general.

2. It is easy to see that (1) (implies and hence) is equivalent to the condition \( \mu \in A^\perp \Rightarrow \mu_F = 0 \). Does the theorem hold if we replace the dominated extension property (1) by the condition \( \mu \in A^\perp \Rightarrow \mu_F \in A^\perp \) and \( C(F) \) by the restriction algebra \( A|F| \)?

**ACKNOWLEDGEMENT.** The author wishes to thank Professor Donald Marshall for a helpful discussion.

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**Department of Mathematics, University of Washington, Seattle, Washington 98195**

**Current address:** Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Ljubljana, Yugoslavia