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ON THE BOUNDARY OF REDUCED TEICHMÜLLER SPACE[1]

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ABSTRACT. Quasifuchsian groups used in the description of the boundary of Teichmüller space do *not* have analogs for reduced Teichmüller space.

Let G be a finitely generated nonelementary Fuchsian group of the second kind, with ordinary set $\Omega = \Omega(G)$. In [2] and [3], Earle describes a map of $T^{\#}(G)$, the reduced Teichmüller space of G, onto an open bounded domain in a Banach space of bounded quadratic differentials for G. This map induces a canonical real analytic structure on $T^{\#}(G)$. Following Bers [1], we discuss the boundary of the image of $T^{\#}(G)$.

Earle's map may be described as follows. Choose a cover map ρ: $U \to \Omega$ satisfying $\rho \circ J(z) = \bar{\rho}(z)$, where $J(z) = -\bar{z}$. (Earle demonstrates in [3] that the analytic structure of $T^{\#}(G)$ is actually independent of this choice of cover.) Let $H = \{h \in PSL(2, \mathbf{R}) | \rho \circ h = g \circ \rho$ for some $g \in G\}$ be the Fuchsian equivalent of G. J induces a symmetry on H, and an anticonformal involution with fixed curves on U/H. There is a natural projection P: $H \to G$, and it is easy to compute that if $P(h) = g$, then $P(\tilde{h}) = P(JhJ) = g$ also.

Define $M_1(H)$ ($M_1(G)$) as the open unit ball in the space of Beltrami differentials of H (G). Set $B_2(H, L)$ as the set of bounded quadratic differentials of H ($\|\phi\| = \sup|\phi(z)(z - \bar{z})^2| < \infty$). Let $B_2(G, \Omega)$ be the set of bounded quadratic differentials for G, real on $\mathbf{R} \cap \Omega$ ($\|\phi\| = \sup|\phi(z)\lambda^{-2}| < \infty$ where λ is the Poincaré metric on Ω induced by ρ).

Let $M_1'(H) = \{\mu \in M_1(H) | \mu(J(z)) = \bar{\mu}(z)\}$ and $B_2'(H, U) = \{\phi \in B_2(H, U) | \phi(J(z)) = \bar{\phi}(z)\}$. Then there are natural isomorphisms $M_1(G) \to M_1'(H)$ given by $\mu \to \rho \cdot \mu = \mu(\rho(z))\overline{\rho'(z)}/\rho'(z)$ and $B_2(G, \Omega) \to B_2'(H, L)$ given by $\phi \to \phi \times \rho$ where $\phi \times \rho(\bar{z}) = \phi(\rho(z))\rho'(z)^2$.

Let w_μ be the unique normalized solution of the Beltrami differential equation $w_{\bar{z}} = \mu w_z$ on $\bar{C}$, where μ has been extended to L by $\mu(\bar{z}) = \bar{\mu}(z)$, $\mu \in M_1(H)$ or $M_1(G)$. Then $T(H)$ ($T^{\#}(G)$) the reduced Teichmüller space is defined as the set of equivalence classes $[w_\mu]$, where $w_\mu \sim w_\nu$ ($\mu \sim \nu$) if and only if $w_\mu = w_\nu$ on the limit set $\Lambda(H)$ of H ($\Lambda(G)$ of G). $T'(H)$ is the set of $[w_\mu]$ which are odd on $\mathbf{R}$. Note that $\mu \in M_1'(H)$ if and only if $w_\mu \circ J = J \circ w_\mu$ [2], and each $[w_\mu] \in T'(H)$ contains at

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least one such w_μ, [5]. Let w^μ be the normalized solution when μ is set equal to zero on L.

Then the following map of $T^\#(G)$ into $B_2(G, \Omega)$ is well defined

$$\begin{array}{ccc} T^\#(G) & \to & T'(H) \\ \downarrow & & \downarrow \\ B_2(G,\Omega) & \to & B_2'(H,L) \subset B_2(H,L) \end{array}$$

by

$$\begin{array}{ccc} [\mu] & \to & [\rho\cdot\mu] \\ \downarrow & & \downarrow \\ \phi & \to & \overline{\phi\times\rho}(\bar{z}) \end{array}$$

where $\overline{\phi \times \rho}(\bar{z}) = \{w^{\rho\cdot\mu}, z\}$, the Schwarzian derivative of $w^{\rho\cdot\mu}$, and horizontal arrows represent real analytic isomorphisms. T' and $T^\#$ will be identified with their images. (Recall that for groups of the second kind, $[\mu] \to \{w^\mu, z\}$ is *not* well defined.)

We will describe how the symmetric structure of H is reflected in the cusps on the boundary of $T'(H) \subset T(H)$, and then show that this description *cannot* be applied to the group G, even through the relationship between $T'(H)$ and $T^\#(G)$ is so close.

As in [1], we form two classes of groups. If $\phi \in B_2(H, L) \cap T(H)$, there exists a quasiconformal map w_ϕ: $C \to C$, which satisfies the differential equation $w_{\bar{z}} = \mu w_z$, $\mu \in M_1(H)$, $\mu = 0$ on L, and $\{w_\phi, z\} = \phi$. By Theorems 1 and 2 of [2], $T' = T \cap B'$; in particular, each $\phi \in T'(H)$ is the Schwarzian derivative of a quasiconformal map w_ϕ with $\mu \in M_1'(H)$. It is clear that if a sequence $\{\phi_n\} \in B_2'(H, L)$ has limit ϕ_∞, then $\phi_\infty \in B_2'(H, L)$ also. An isomorphism χ_ϕ is defined for $h \in H$ by $w_\phi(h(z)) = \chi_\phi(h) \circ w_\phi(z)$. The isomorphism $\chi_\phi(H)$ is the first group.

The second class of groups is defined as follows. The quasifuchsian group $\chi_\phi(H)$, $\phi \in T(H)$, represents two Fuchsian groups obtained by conjugating $\chi_\phi(H)$ by conformal maps of $w_\phi(U)$ and $w_\phi(L)$ onto U and L respectively. For the lower map, send $w_\phi(0)$, $w_\phi(1)$ and $w_\phi(\infty)$ onto 0, 1, ∞ respectively; then the lower image is H. We normalize the map ω: $w_\phi(U) \to U$ so that $\omega \circ w$ fixes 0, 1, ∞; then $\hat{\chi}_\phi(H) = H_\mu = (\omega \circ w) \circ H \circ (\omega \circ w)^{-1}$; $\hat{\chi}_\phi$ depends only on ϕ, and not on the choice of μ. Note that H_μ is precisely the group obtained by conjugating H by the unique solution w_μ of the differential equation $w_{\bar{z}} = \mu w_z$; mapping $U \to U$, fixing 0, 1, ∞.

LEMMA 1. *Let the closed curve C on U/H determine the conjugacy class of the element h of H. Then $\tilde{C}$, the curve symmetric to C, determines the conjugacy class of the element $\tilde{h} = J \circ h \circ J$ of H.*

We note that C and $\tilde{C}$ are freely homotopic if and only if they are also freely homotopic to a boundary curve of U/G; in this case all three will determine conjugate elements of H.

LEMMA 2. *Suppose* $h \in H$ *is such that* $\lim_{j\to\infty}$ $(\text{trace } \hat{\chi}_{\phi_j}(h))^2 = 4$, $\phi_j \in T'(H)$. *Then* $\lim_{j\to\infty}$ $(\text{trace } \hat{\chi}_{\phi_j}(\tilde{h}))^2 = 4$ *also.*

PROOF. Since each $\phi_j \in T'(H)$, we may choose a representative w_ϕ where w_ϕ has dilatation μ on U, $\mu \in M_1'(H)$. Then w_μ commutes with J, and

$$\hat{\chi}_{\phi_j}(\tilde{h}) = w_\mu \circ \tilde{h} \circ w_\mu^{-1} = w_\mu \circ J \circ h \circ J \circ w_\mu^{-1}$$
$$= J \circ w_\mu \circ h \circ w_\mu^{-1} \circ J = J \circ \hat{\chi}_{\phi_j}(h) \circ J.$$

Hence $(\text{trace } \hat{\chi}_{\phi_j}(h))^2 = (\text{trace } \hat{\chi}_{\phi_j}(\tilde{h}))^2$ for each j.

From Lemma 1 we conclude that "pinching" one loop via a ϕ, $\phi \in T'(H)$, induces a "pinching" of the symmetric loop. We next demonstrate the existence of appropriate ϕ_j's.

PROPOSITION. *Let* $\hat{C} = \{C_1, C_2, \ldots, C_N\}$ *be a set of disjoint simple loops on* U/H, *no two of which are freely homotopic, and such that if* $C_k \in C$, *then the curve symmetric to* C_k, $\tilde{C}_k \in \hat{C}$. *Let the elements* h_k *determined by the* C_k *be hyperbolic. Then there is a mapping* $t \to \phi_t$ *of* $0 < t < \infty$ *into* $T'(H)$ *such that*

$$\lim_{t\to\infty} \left(\text{trace } \hat{\chi}_{\phi_t}(h_k)\right)^2 = \lim_{t\to\infty} \left(\text{trace } \hat{\chi}_{\phi_t}(\tilde{h}_k)\right)^2 = 4, \qquad k = 1, \ldots, N;$$

hence there are cusps on the boundary of $T'(H)$.

PROOF. We assume that $\hat{C}$ contains one pair of symmetric loops, C and $\tilde{C}$, and leave the general case to the reader. Suppose C is not homotopic to a curve fixed by the anticonformal involution of U/H. Then C is distinct from and not homotopic to $\tilde{C}$. Follow the construction of Bers [1, Theorem 11]. Replace an annulus D, $0 < a < |\xi| < 1$ with C homotopic to $|\xi| = \sqrt{a}$, by the annulus D_t, $0 < a^t < |\xi| < 1$. Similarly replace the annulus $\tilde{D}$ symmetric to D, by $\tilde{D}_t$, symmetric to D_t. If C is homotopic to a boundary curve, it is also homotopic to $\tilde{C}$; hence $C = \tilde{C}$. Then C may be bordered by two symmetric annuli A_1 and A_2, $0 < a < |\xi| < 1$ and $0 < a < |\xi'| < 1$, respectively. Let $D = A_1 \cup A_2$ identified along the curves $|\xi| = |\xi'| = a$. We may replace the annulus D by the annulus D_t obtained by replacing A_i by the symmetric annuli A_{it}, $0 < a^t < |\xi| < 1$, $i = 1, 2$, again properly identified along the boundaries. Then the module of D_t is equal to twice the module of A_{it}, and hence becomes infinite as t goes to infinity.

We let $S^{(t)}$ be the symmetric surface obtained by replacing each D by the corresponding D_t. Then there is a map f_t: $U/H = S \to S^{(t)}$, respecting the symmetries, which is the identity on $S - \{D\}$ and maps a point $re^{i\theta}$ of $D(A_i)$ onto $r^te^{i\theta}$ of $D_t(A_{it})$. The anticonformal involution σ_t on $S^{(t)}$ lifts to U; replacing the universal cover π by $\pi \circ A$, $A \in PSL(2, \mathbf{R})$, if necessary, we may assume this lift is J, [2]. Hence we may lift f_t to a map w_t: $U \to U$ such that $w_t \circ J = J \circ w_t$, and the group $H_t = w_t \circ H \circ w_t^{-1}$ is a Fuchsian group representing the symmetric surface $S^{(t)} = U/H_t$.

For each t, there is a $\phi_t \in T(H)$ such that $\hat{\chi}_{\phi_t}(h) = w_t \circ h \circ w_t^{-1}$. If $w_t = w_\mu$ commutes with J, $\{w^\mu, z\} = \phi \in B_2'(H, U)$, and so ϕ_t is actually in $T'(H)$.

The rest of the proof follows as in Theorems 11 and 12 of [1], since the moduli of the various D_t become infinite, and the limit of a sequence of elements of $T'(H)$ is also symmetric.

We next attempt to determine how the relationships of H, $\chi(H)$ and $\hat{\chi}(H)$ apply to the group G.

For $\mu \in M_1(G)$, w_μ satisfies $w_\mu(\bar{z}) = \overline{w_\mu(z)}$. Then the normalized mapping $w_{\rho\cdot\mu}$ has the property that $w_\mu \circ \rho \circ (w_{\rho\cdot\mu})^{-1} = \rho_\mu$ is a holomorphic cover map $U \to \Omega(G_\mu)$. Now $\rho_\mu \circ J = \bar{\rho}_\mu$, and $H_{\rho\cdot\mu}$ is the Fuchsian equivalent of G_μ. Further, since $w_{\rho\cdot\mu}$ commutes with J, both $w_{\rho\cdot\mu} \circ h \circ (w_{\rho\cdot\mu})^{-1}$ and $w_{\rho\cdot\mu} \circ \tilde{h} \circ (w_{\rho\cdot\mu})^{-1} = J \circ w_{\rho\cdot\mu} \circ h \circ (w_{\rho\cdot\mu})^{-1} \circ J$ will project to a given $g \in G_\mu$.

Theorem. *Any quasifuchsian analog of $\chi(H)$ is actually Fuchsian.*

Proof. Suppose we have defined some quasifuchsian group $\chi_\phi(G)$, such that $\chi_{\phi\times\rho}(H) = \{\gamma | \eta \circ \pi = \pi \circ \gamma \text{ for some } \eta \in \chi_\phi(G)\}$ where π is a cover map from the simply connected region $w^{\rho\cdot\mu}(L)$ onto $\Omega(\chi_\phi(G))$. By the definitions of H and $H_{\rho\cdot\mu} = \hat{\chi}_{\phi\times\rho}(H)$, the elements h and $\tilde{h} = JhJ$ are paired in the projection to G, and $h_{\rho\cdot\mu}$ and $\tilde{h}_{\rho\cdot\mu}$ are paired in the projection to G_μ. Hence for a well-defined $\chi_\phi(G)$, $h^\mu = \chi_{\phi\times\rho}(h)$ and $\tilde{h}^\mu = \chi_{\phi\times\rho}(\tilde{h})$ should project to one element of $\chi_\phi(G)$. Since $T' = T \cap B'$ and $\{w^{\rho\cdot\mu}, z\} \in B'(H, L)$ then $w^{\rho\cdot\mu} \circ J \circ (w^{\rho\cdot\mu})^{-1} = \bar{K}$ is the conjugate of an involutory fractional linear transformation on $w^{\rho\cdot\mu}(L)$, [2]. But then

$$\begin{aligned}\chi(\tilde{h}) &= w^{\rho\cdot\mu} \circ J \circ h \circ J \circ (w^{\rho\cdot\mu})^{-1} \\ &= \bar{K} \circ w^{\rho\cdot\mu} \circ h \circ (w^{\rho\cdot\mu})^{-1} \circ (\bar{K})^{-1} = \bar{K}\chi(h)(\bar{K})^{-1}\end{aligned}$$

and $\bar{K}$ induces a well-defined automorphism of any subgroup of $\chi(H)$ containing both elements to be paired, in particular, of the cover group of π. Then $\bar{K}$ must project to an anticonformal involution $\bar{k}$ of $\Omega(\chi_\phi(G))$. But if $g \circ \pi = \pi \circ \chi(h) = \pi \circ \chi(\tilde{h})$, then

$$\bar{k} \circ g \circ \pi = \bar{k} \circ \pi(\chi(\gamma)) = \pi \circ \bar{K} \circ \chi(\gamma) = \pi \circ \chi(\tilde{\gamma}) \circ \bar{K} = g \circ \pi \circ \bar{K} = g \circ \bar{k} \circ \pi,$$

and since π is a cover map, $\bar{k} \circ g = g \circ \bar{k}$ on $\Omega(\chi_\phi(G))$. But then by [4], $\bar{k}$ is the restriction of an anticonformal involutory fractional linear transformation, and $\chi_\phi(G)$ is in fact Fuchsian.

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