ON THE BOUNDARY OF REDUCED
TEICHMÜLLER SPACE

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ABSTRACT. Quasifuchsian groups used in the description of the boundary of
Teichmüller space do not have analogs for reduced Teichmüller space.

Let G be a finitely generated nonelementary Fuchsian group of the second kind,
with ordinary set Ω = Ω(G). In [2] and [3], Earle describes a map of \( T^d(G) \), the
reduced Teichmüller space of G, onto an open bounded domain in a Banach space
of bounded quadratic differentials for G. This map induces a canonical real
analytic structure on \( T^d(G) \). Following Bers [1], we discuss the boundary of the
image of \( T^d(G) \).

Earle's map may be described as follows. Choose a cover map \( ρ: U \to Ω \)
satisfying \( ρ \circ J(z) = \bar{ρ}(z) \), where \( J(z) = -z \). (Earle demonstrates in [3] that the
analytic structure of \( T^d(G) \) is actually independent of this choice of cover.) Let
\( H = \{ h \in PSL(2, \mathbb{R}) | ρ \circ h = g \circ ρ \text{ for some } g \in G \} \) be the Fuchsian equivalent
of G. J induces a symmetry on H, and an anticonformal involution with fixed
curves on \( U/H \). There is a natural projection \( P: H \to G \), and it is easy to compute
that if \( P(h) = g \), then \( P(\bar{h}) = P(JhJ) = g \) also.

Define \( M_1(H) \) (\( M_1(G) \)) as the open unit ball in the space of Beltrami differen-
tials of \( H \) (\( G \)). Set \( B_2(H, L) \) as the set of bounded quadratic differentials for G, real on \( R \cap Ω \) \(|\phi| = \sup|\phi(z)(z - \bar{z})^2| < \infty \) where \( λ \) is the Poin-
caré metric on Ω induced by \( ρ \).

Let \( M_1(H) = \{ μ \in M_1(H) | μ(J(z)) = \bar{μ}(z) \} \) and \( B_2(H, U) = \{ φ \in B_2(H, U) | φ(J(z)) = φ(z) \} \). Then there are natural isomorphisms \( M_1(G) \to M_1(H) \)
given by \( μ \mapsto ρ \circ μ = \bar{μ}(ρ(z))\bar{ρ}'(z)/ρ'(z) \) and \( B_2(G, Ω) \to B_2(H, L) \) given by \( φ \mapsto φ \times ρ \) \(|φ × ρ(\bar{z}) = φ(ρ(z))ρ'(z)^2| \).

Let \( w_μ \) be the unique normalized solution of the Beltrami differential equation
\( w_μ = μw_μ \) on \( C \), where \( μ \) has been extended to \( L \) by \( μ(\bar{z}) = \bar{μ}(z) \), \( μ \in M_1(H) \) or
\( M_1(G) \). Then \( T(H) \) (\( T^d(G) \)) the reduced Teichmüller space is defined as the set of
equivalence classes \([w_μ]\), where \( w_μ \sim w_ν \) if and only if \( w_μ \sim w_ν \) on the limit
set \( Δ(H) \) of H (\( Δ(G) \) of G). \( T'(H) \) is the set of \([w_μ]\) which are odd on \( R \). Note that
\( μ \in M_1(H) \) if and only if \( w_μ \circ J = J \circ w_μ \) [2], and each \([w_μ]\) \( ∈ T'(H) \) contains at
least one such $w_{\mu}$, [5]. Let $w_{\mu}$ be the normalized solution when $\mu$ is set equal to zero on $L$.

Then the following map of $T^g(G)$ into $B_2(G, \Omega)$ is well defined

$$
\begin{array}{ccc}
T^g(G) & \rightarrow & T'(H) \\
\downarrow & & \downarrow \\
B_2(G, \Omega) & \rightarrow & B_2(H, L) \subset B_2(H, L)
\end{array}
$$

by

$$
\begin{array}{rcl}
\begin{bmatrix} \mu \end{bmatrix} & \rightarrow & \begin{bmatrix} \rho \cdot \mu \end{bmatrix} \\
\phi & \rightarrow & \phi \times \rho(\bar{z})
\end{array}
$$

where $\phi \times \rho(\bar{z}) = \{w^{\rho \mu}, z\}$, the Schwarzian derivative of $w^{\rho \mu}$, and horizontal arrows represent real analytic isomorphisms. $T'$ and $T^g$ will be identified with their images. (Recall that for groups of the second kind, $[\mu] \rightarrow \{w_{\mu}, z\}$ is not well defined.)

We will describe how the symmetric structure of $H$ is reflected in the cusps on the boundary of $T'(H) \subset T(H)$, and then show that this description cannot be applied to the group $G$, even through the relationship between $T'(H)$ and $T^g(G)$ is so close.

As in [1], we form two classes of groups. If $\phi \in B_2(H, L) \cap T(H)$, there exists a quasiconformal map $w_{\phi}: C \rightarrow C$, which satisfies the differential equation $w_z = \mu w_z$, $\mu \in M_1(H)$, $\mu = 0$ on $L$, and $\{w_{\phi}, z\} = \phi$. By Theorems 1 and 2 of [2], $T = T \cap B'$; in particular, each $\phi \in T'(H)$ is the Schwarzian derivative of a quasiconformal map $w_{\phi}$ with $\mu \in M_1(H)$. It is clear that if a sequence $\{\phi_n\} \in B_2(H, L)$ has limit $\phi_\infty$, then $\phi_\infty \in B_2(H, L)$ also. An isomorphism $\chi_{\phi}$ is defined for $h \in H$ by $w_{\phi}(h(z)) = \chi_{\phi}(h) \circ w_{\phi}(z)$. The isomorphism $\chi_{\phi}(H)$ is the first group.

The second class of groups is defined as follows. The quasifuchsian group $\chi_{\phi}(H)$, $\phi \in T(H)$, represents two Fuchsian groups obtained by conjugating $\chi_{\phi}(H)$ by conformal maps of $w_{\phi}(U)$ and $w_{\phi}(L)$ onto $U$ and $L$ respectively. For the lower map, send $w_{\phi}(0)$, $w_{\phi}(1)$ and $w_{\phi}(\infty)$ onto 0, 1, $\infty$ respectively; then the lower image is $H$. We normalize the map $\omega: w_{\phi}(U) \rightarrow U$ so that $\omega \circ w$ fixes 0, 1, $\infty$; then $\hat{\chi}_{\phi}(H) = H_\mu = (\omega \circ w) \circ H \circ (\omega \circ w)^{-1}$; $\hat{\chi}_{\phi}$ depends only on $\phi$, and not on the choice of $\mu$. Note that $H_\mu$ is precisely the group obtained by conjugating $H$ by the unique solution $w_{\mu}$ of the differential equation $w_z = \mu w_z$; mapping $U \rightarrow U$, fixing $0, 1, \infty$.

**Lemma 1.** Let the closed curve $C$ on $U/H$ determine the conjugacy class of the element $h$ of $H$. Then $\tilde{C}$, the curve symmetric to $C$, determines the conjugacy class of the element $\tilde{h} = J \circ h \circ J$ of $H$.

We note that $C$ and $\tilde{C}$ are freely homotopic if and only if they are also freely homotopic to a boundary curve of $U/G$; in this case all three will determine conjugate elements of $H$. 

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Lemma 2. Suppose \( h \in H \) is such that \( \lim_{j \to \infty} (\text{trace } \hat{x}_\phi(h))^2 = 4 \), \( \phi_j \in T'(H) \). Then \( \lim_{j \to \infty} (\text{trace } \hat{x}_\phi(h))^2 = 4 \) also.

Proof. Since each \( \phi_j \in T'(H) \), we may choose a representative \( w_\phi \) where \( w_\phi \) has dilatation \( \mu \) on \( U \), \( \mu \in M'(H) \). Then \( w_\phi \) commutes with \( J \), and

\[
\hat{x}_\phi(h) = w_\phi \circ h \circ w_\phi^{-1} = w_\phi \circ J \circ h \circ J \circ w_\phi^{-1} = J \circ w_\phi \circ h \circ w_\phi^{-1} \circ J = J \circ \hat{x}_\phi(h) \circ J.
\]

Hence \( (\text{trace } \hat{x}_\phi(h))^2 = (\text{trace } \hat{x}_\phi(h))^2 \) for each \( j \).

From Lemma 1 we conclude that "pinching" one loop via a \( \phi, \phi \in T'(H) \), induces a "pinching" of the symmetric loop. We now demonstrate the existence of appropriate \( \phi_j \)'s.

Proposition. Let \( C = \{C_1, C_2, \ldots, C_N\} \) be a set of disjoint simple loops on \( U/H \), no two of which are freely homotopic, and such that if \( C_k \in C \), then the curve symmetric to \( C_k \), \( \tilde{C}_k \in \tilde{C} \). Let the elements \( h_k \) determined by the \( C_k \) be hyperbolic. Then there is a mapping \( t \to \phi_t \) of \( 0 < t < \infty \) into \( T'(H) \) such that

\[
\lim_{t \to \infty} (\text{trace } \hat{x}_\phi(h_k))^2 = \lim_{t \to \infty} (\text{trace } \tilde{x}_\phi(h_k))^2 = 4, \quad k = 1, \ldots, N;
\]

hence there are cusps on the boundary of \( T'(H) \).

Proof. We assume that \( \tilde{C} \) contains one pair of symmetric loops, \( C \) and \( \tilde{C} \), and leave the general case to the reader. Suppose \( C \) is not homotopic to a curve fixed by the anticonformal involution of \( U/H \). Then \( C \) is distinct from and not homotopic to \( \tilde{C} \). Follow the construction of Bers [1, Theorem 11]. Replace an annulus \( D, 0 < a < |\xi| < 1 \) with \( C \) homotopic to \( |\xi| = \sqrt{a} \), by the annulus \( D_a, 0 < a' < |\xi| < 1 \). Similarly replace the annulus \( D \) symmetric to \( D \), by \( D_{a'} \) symmetric to \( D \). If \( C \) is homotopic to a boundary curve, it is also homotopic to \( \tilde{C} \); hence \( C = \tilde{C} \). Then \( C \) may be bordered by two symmetric annuli \( A_1 \) and \( A_2, 0 < a < |\xi| < 1 \) and \( 0 < a < |\xi'| < 1 \), respectively. Let \( D = A_1 \cup A_2 \) identified along the curves \( |\xi| = |\xi'| = a \). We may replace the annulus \( D \) by the annulus \( D_t \), obtained by replacing \( A_i \) by the symmetric annuli \( A_{i'}, 0 < a' < |\xi| < 1, i = 1, 2 \), again properly identified along the boundaries. Then the module of \( D_t \) is equal to twice the module of \( A_{i'} \), and hence becomes infinite as \( t \) goes to infinity.

We let \( S^{(i)} \) be the symmetric surface obtained by replacing each \( D \) by the corresponding \( D_t \). Then there is a map \( f : U/H = S \to S^{(i)} \), respecting the symmetries, which is the identity on \( S - \{D\} \) and maps a point \( r \) of \( D(A) \) onto \( r \) of \( D_t(A_t) \). The anticonformal involution \( \sigma \) on \( S^{(i)} \) lifts to \( U \); replacing the universal cover \( \pi \) by \( \pi \circ A, A \in \text{PSL}(2, \mathbb{R}), \) if necessary, we may assume this lift is \( J \), [2]. Hence we may lift \( f \) to a map \( w : U \to U \) such that \( w_t \circ J = J \circ w_t \), and the group \( H = w_t \circ H \circ w_t^{-1} \) is a Fuchsian group representing the symmetric surface \( S^{(i)} = U/H_t \).

For each \( t \), there is a \( \phi_t \in T(H) \) such that \( \hat{x}_\phi(h) = w_t \circ h \circ w_t^{-1} \). If \( w_t = w_\phi \) commutes with \( J \), \( \{w_\phi, z\} = \phi \in B_2(H, U), \) and so \( \phi_t \) is actually in \( T'(H) \).
The rest of the proof follows as in Theorems 11 and 12 of [1], since the moduli of the various \( D_i \) become infinite, and the limit of a sequence of elements of \( T'(H) \) is also symmetric.

We next attempt to determine how the relationships of \( H, \chi(H) \) and \( \hat{\chi}(H) \) apply to the group \( G \).

For \( \mu \in M_f(G), w_\mu \) satisfies \( w_\mu(\bar{z}) = \overline{w_\mu(z)} \). Then the normalized mapping \( w_{p_\mu} \) has the property that \( w_{p_\mu} \circ \rho \circ (w_{p_\mu})^{-1} = \rho_\mu \) is a holomorphic cover map \( U \rightarrow \Omega(G_\mu) \). Now \( \rho_\mu \circ J = \overline{\rho_\mu} \), and \( H_{p_\mu} \) is the Fuchsian equivalent of \( G_\mu \). Further, since \( w_{p_\mu} \) commutes with \( J \), both \( w_{p_\mu} \circ h \circ (w_{p_\mu})^{-1} \) and \( w_{p_\mu} \circ \hat{h} \circ (w_{p_\mu})^{-1} = J \circ w_{p_\mu} \circ h \circ (w_{p_\mu})^{-1} \circ J \) will project to a given \( g \in G_\mu \).

**Theorem.** Any quasifuchsian analog of \( \chi(H) \) is actually Fuchsian.

**Proof.** Suppose we have defined some quasifuchsian group \( \chi_\phi(G) \), such that \( \chi_{\phi \times \chi}(H) = \{ Y | \eta \circ \pi = \pi \circ \gamma \) for some \( \eta \in \chi_\phi(G) \} \) where \( \pi \) is a cover map from the simply connected region \( w_{p_\mu}(L) \) onto \( \Omega(\chi_\phi(G)) \). By the definitions of \( H \) and \( H_{p_\mu} = \chi_{\phi \times \chi}(H) \), the elements \( h \) and \( \hat{h} = JhJ \) are paired in the projection to \( G \), and \( h_{p_\mu} \) and \( \hat{h}_{p_\mu} \) are paired in the projection to \( G_\mu \). Hence for a well-defined \( \chi_\phi(G) \), \( h^p = \chi_{\phi \times \chi}(h) \) and \( \hat{h}^p = \chi_{\phi \times \chi}(\hat{h}) \) should project to one element of \( \chi_\phi(G) \). Since \( T' = T \cap B' \) and \( \{ w_{p_\mu}, z \} \in \beta(B'(H, L)) \) then \( w_{p_\mu} \circ J \circ (w_{p_\mu})^{-1} = \overline{K} \) where \( \overline{K} \) is the conjugate of an involutory fractional linear transformation on \( w_{p_\mu}(L) \), [2]. But then

\[
\chi(\hat{h}) = w_{p_\mu} \circ J \circ h \circ J \circ (w_{p_\mu})^{-1}
\]

and \( \overline{K} \) induces a well-defined automorphism of any subgroup of \( \chi(H) \) containing both elements to be paired, in particular, of the cover group of \( \pi \). Then \( \overline{K} \) must project to an anticonformal involution \( \hat{K} \) of \( \Omega(\chi_\phi(G)) \). But if \( g \circ \pi = \pi \circ \chi(h) = \pi \circ \chi(\hat{h}) \), then

\[
\hat{K} \circ g \circ \pi = \hat{K} \circ \pi(\gamma) = \pi \circ \hat{K} \circ \gamma(\chi) = \pi \circ \chi(\hat{y}) \circ \hat{K} = g \circ \pi \circ \hat{K} = g \circ \hat{K} \circ \pi,
\]

and since \( \pi \) is a cover map, \( \hat{K} \circ g = g \circ \hat{K} \) on \( \Omega(\chi_\phi(G)) \). But then by [4], \( \hat{K} \) is the restriction of an anticonformal involutory fractional linear transformation, and \( \chi_\phi(G) \) is in fact Fuchsian.

**References**


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